A New Fractional Order Hyper-Chaotic System Design and its Control Circuit Simulation

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Abstract: Fractional order chaotic system has more complex features than integer order chaotic systems. In this paper, a new fractional order hyper-chaotic system was constructed, whose dynamic properties were analyzed by software simulation \( q = 0.80 - 0.99 \) (step size is 0.01). Based on the frequency domain Bode plot approximation, a new system circuit was designed by tree circuit unit. The circuit simulation and the corresponding test show the same results with simulation using software, whereby the availability of the circuit was described. Finally, a simple linear feedback controller was designed for circuit simulation; the simulation results also verify the feasibility of the controller.

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Keywords: Fractional order, Hyper-chaotic, Frequency domain approximation method, Circuit simulation, Linear feedback.

1. Introduction

Since 1963, Lorenz found chaos when he studied at atmospheric temperature convection [1], characteristics of chaotic nonlinear dynamics have been come through in integer order systems such as Chua's circuit, Chen system, Duffing system, hyper-chaotic system [2, 3], and so on. Especially in the two aspects of control and synchronization, they have achieved good results [4, 5]. Fractional differentiation is an extension of integer-order theory of differentiation. Although research on fractional order differentiation has three centuries, and correspond to the history of integer order differentiation, but it was limited due to the lack of practical application. In recent years, when researchers introduced fractional order differential operator to chaotic system, they found that it still can show the wonderful chaotic or hyper-chaotic phenomena [6, 7], and better described by the characteristics of chaotic system. From then on, fractional order chaotic systems have begun to appear the new wave of research in physics and engineering fields [8-10].

Due to complexity of high dimensional fractional order chaotic system, the tracking reports on synchronization control of high dimensional fractional order chaotic systems is less, This paper we construct a new four-dimensional fractional hyper-chaotic system, analysis its balance and stability and the other related dynamics, using Matlab for its numerical simulation, adopting differentiation method of approximate theory and potter figure in the frequency domain, designing fractional order tree circuit unit [11, 12], at the same time applying Multisim software to simulation implementation. To achieve the synchronization control of the system, a simple linear feedback controller is proposed, numerical calculation and circuit simulation verify the effectiveness.
### 2. Fractional Differentiation and its Frequency-Domain Approximation

Fractional order differentiation has many definitions [13], commonly definition is Riemann Liouville-(RL) is defined, its mathematical expression is as follows:

\[
\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \left[ \frac{f(\tau)}{(t-\tau)^{q-n+1}} \right]
\]  
\tag{1}

among eq.(1), \( \Gamma() \) is the gamma function, \( n-1<q<n \), \( q \) is the fraction, \( n \) is the integer, eq.(1) is the unified representation of the fractional order differentiation and the fractional order integration. It shows that the fractional order differentiation is mnemonic function. so, fractional order differentiation is more suitable for the description with circuit characteristics. If the initial value of the function \( f(t) \) is zero, the Laplace transform formula of eq. (1) can be expressed as:

\[
L\left[ \frac{d^q f(t)}{dt^q} \right] = s^q L[f(t)]
\]  
\tag{2}

Therefore, fractional order differential operator \( q \) can be expressed in transfer function of the frequency domain \( H(s) = \frac{1}{s^q} \). Refer to literature [14] which proposed a frequency domain approximate method, an approximate formula \( \frac{1}{s^q} \) (\( q = 0.1-0.9 \), step is 0.1) was derived. In this article we only use approximate error is 2 dB of approximate formula.

### 2. A New Fractional Order Hyper-Chaotic System

Based on a three-dimensional constant order chaos equation, a new four-dimensional fractional hyper-chaotic system is designed; its dynamic equation is as follows:

\[
\begin{align*}
\frac{d^aq}{dt^q} x &= a(y-x) + w \\
\frac{d^aq}{dt^q} y &= by - 4xz^2 \\
\frac{d^aq}{dt^q} z &= xyz - hz + cy \\
\frac{d^aq}{dt^q} w &= 10z
\end{align*}
\]  
\tag{3}

where \( a=10, b=5, c=0.5, h=5 \). By eq. (3) we can get the balance plots \( O(0,0,0,0) \) of the system, and system only has a balance point, it is original point. Because in position of balance point:

\[
\nabla V = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} + \frac{\partial w}{\partial w}
= -a + b + h = -10 < 0
\]  
\tag{4}

Thus the system is dissipative, and convergence with exponent \( \frac{dV}{dt} = e^{-10} \). As time goes on, the system is asymptotically tends to an attractor.

In order to more intuitive understand the characteristics of the hyper-chaos system dynamics, using Matlab software to numerical simulation of the system (3). Choosing initial values for the system \((x_0, y_0, z_0, w_0) = (2, 1, 3, 2)\), when \( q = 0.90 \), fractional order chaotic system attractor is shown in Fig. 1.

\[\text{Fig. 1. Dynamics trajectory of a new chaotic system (} q=0.90).\]

Fig. 1 shows the fractional order hyper-chaotic system has continuous aperiodicity, and it is very sensitive to the initial value, and has more than 1 greater than zero Lyapnuov exponent, which is hyper-chaotic system to produce a fundamental nature.

By changing \( q \) value, we can simulate different phase diagrams when \( q = 0.50 \), \( q = 0.66 \) and \( q = 0.79 \). The system can not produce chaos (see Fig. 2 (a)-(c)), when \( q = 0.80 \), in this critical value, system begin to appear chaotic attractor. In fact, through simulating a large number of software, equation (3) is suitable to all \( q = 0.80 - 0.99 \) (step is 0.01) fractional order numerical range.
A and B it can be achieved reference [14] $1/s^q$ approximation from 0.1 to 0.9, in order to convenient, we call it a tree circuit unit.

![Diagram of a tree circuit unit](image)

As an example for $q = 0.90$, design circuit simulation with system (3), according to the reference [14], we know the approximate equation (2 dB) of $1/s^q$ is:

$$
\frac{1}{s^{0.9}} \approx \frac{2.2675(s + 1.292)(s + 215.4)}{(s + 0.01292)(s + 2.154)(s + 359.4)}
$$

When $q = 0.90$, tree unit circuit is shown in Fig. 4:

![Diagram of a tree circuit unit](image)

Fractional order transfer function is:

$$
H(s) = \left( R_1 + \frac{1}{sC_2} \right) \left( \frac{1}{s^2C_1} + \left( \frac{1}{sC_1} + \frac{1}{sC_3} \right) \right)
$$

$$
= \frac{1}{C_0} \left( \frac{C_0}{C_1} R_1 + \frac{C_3}{C_1} R_2 \right) \left( \frac{C_0}{C_3} R_1 \right) \left( \frac{1}{sC_1} + \frac{1}{sC_3} \right)
$$

$$
= \frac{R_1 + R_2}{C_2 R_1 R_2} + \frac{1}{C_3 R_1 R_2} + \frac{C_1 + C_3}{C_1 C_2 C_3 R_1 R_2} \left( \frac{R_3 + R_4}{C_3 R_3 R_4} \right) s + \frac{1}{C_1 C_2 C_3 R_1 R_2}
$$

4. Hyper-Chaotic System Circuit Design and Simulation Analysis

4.1. Analysis of Fractional Order Tree Circuit Unit

According to the theory of circuit, Fig. 3 is equivalent circuit of the frequency domain, between

Fig. 2. Comparing $x - y$ phase diagram with different $q$ values.

3. The Stability Theory of Fractional Order Linear System

Lemma [13]: considering fractional order autonomous system:

$$
D^q = AX, \quad X(0) = X_0
$$

Among them, $0 < q < 1$, $X \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$.

1) The system is asymptotically stable, if and only if for any of eigenvalues $\lambda$, there all have

$$
\arg(\lambda) > \frac{\pi}{2}
$$

set up.

2) The system is stable, and only if for any eigenvalues $\lambda$, they all having

$$
\arg(\lambda) > \frac{\pi}{2}
$$

set up.

Therefore, for the number of fractional order system $q (0 < q < 1)$, no matter what the value of the state variables, as long as the characteristic root of the system is real component, and is greater than zero, then the fractional order system is asymptotically stable.

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![Diagram of a tree circuit unit](image)
Among them, $C_0$ as unit parameter, and make $C_0=1\mu F$, $F(s)=H(s)$, $C_0=1/s^{0.90}$ compare with eq. (6) and eq. (7), Fig. 4 showing resistor and capacitor values of circuit unit, that are $R_1=1.55\; M\Omega$, $R_2=61.54\; M\Omega$, $R_3=2.526\; M\Omega$, $C_1=0.7346\; \mu F$, $C_2=0.5221\; \mu F$, $C_3=1.103\; \mu F$. Considering the actual resistance capacitance can’t meet the required accuracy, we select $R_1=1.55\; M\Omega$, $R_2=62\; M\Omega$, $R_3=2.52\; M\Omega$, $C_1=0.72\; F$, $C_2=0.522\; F$, $C_3=1.1\; F$ to simulate this experiment.

4.2. Circuit Design of a New Fractional Order Hyper-chaotic System

According to the new design of the fractional order hyper-chaotic system (eq. 3), using Multisim software, we design this circuit with TL082 series of operational amplifier, AD63 analog multiplier, linear resistor and capacitor, etc., and successfully achieve the fractional order operation in tree unit circuit $1/s^{q}$. Fig. 5 shows the detail circuit with $q=0.90$, and these resistances are matched on the respective components.
Let $x = 10u_1$, $y = 10u_2$, $z = 10u_3$, $w = 10u_4$, $\tau = \frac{t}{100RC}$, then eq.(3) is changed by:

$$
\frac{d^q u_1}{d\tau^q} = \frac{1}{RC} R_1 u_1, \quad \frac{d^q u_2}{d\tau^q} = \frac{1}{RC} R_2 u_2, \quad \frac{d^q u_3}{d\tau^q} = \frac{1}{RC} R_3 u_3, \quad \frac{d^q u_4}{d\tau^q} = \frac{1}{RC} R_4 u_4,
$$

(1)

Acturally, eq.(8) and eq.(3) are equivalent.

Simulating the new hyper-chaotic system in Multisim, and the results are shown in Fig. 6. We can clearly see that the results are almost the same as the results in Matlab.

![Phase diagrams](image)

(a) $x-y$ phase diagram; (b) $x-z$ phase diagram;

Fig. 6. Phase diagram of circuit simulation in Multisim.

By changing the fractional order $q$ value of the tree circuit unit in Fig. 3, we can also simulate other fractional order hyper-chaotic circuit with $q = 0.80 - 0.99$ (step is 0.01). Through circuit simulation and numerical simulation that we can get a similar conclusion: the new fractional order nonlinear equations which are proposed in this paper can generate chaotic phenomenon, and has the common characteristic of the hyper-chaos system.

4.3. Feedback Control of the New Fractional Order Chaotic System

The system of linear feedback term, its stable in equilibrium point of the system, which the controlled system can be expressed as:

$$
\begin{align*}
\frac{d^q x}{d\tau^q} &= a(y - x) + w - k_1 x \\
\frac{d^q y}{d\tau^q} &= b y - 4x^2 - k_2 x \\
\frac{d^q z}{d\tau^q} &= xyz - h z + cy - k_3 z \\
\frac{d^q w}{d\tau^q} &= 10z - k_4 w
\end{align*}
$$

(9)

Obviously origin point $(0,0,0,0)$ is the only equilibrium point of the system, then linearize system (9) at the origin point, and get the matrix of Jacobi system as follows:

$$
J(0) = 
\begin{bmatrix}
-a - k_1 & 10 & 0 & 1 \\
0 & b - k_2 & 0 & 0 \\
0 & 0.5 & -h - k_3 & 0 \\
0 & 0 & 10 & -k_4
\end{bmatrix}
$$

(10)

According to $\det[\lambda I - J(0)] = 0$, getting the characteristic root of system:

$$
\lambda_1 = -a - k_1, \quad \lambda_2 = b - k_2, \quad \lambda_3 = -h - k_3, \quad \lambda_4 = -k_4.
$$

For $0 < q < 1$, no matter what the value of the state variable is, as long as the real part of the characteristic roots of the controlled system (9) are not greater than zero, so the controlled system (9) can be asymptotically to the equilibrium point, next we can get:

$$
\begin{align*}
k_1 + a &\geq 0 \\
k_2 - b &\geq 0 \\
k_3 + h &\geq 0 \\
k_4 &\geq 0
\end{align*}
$$

(11)

When $a = 10, b = 5, c = 0.5, h = 5$, we can achieve $k_1 = 0, k_2 = 50, k_3 = 2, k_4 = 0$, and control the new fractional order hyper-chaotic system (9), then the linear feedback control circuit system is shown in Fig. 7:

![Feedback controller](image)

Fig. 7. Linear feedback controller.
If we add the linear feedback controller into the Fig. 5, when the switch J1 and J2 are closed, feedback controller begins to work. Fig. 8 shows the stabilize waveforms of different variables in the controlled system.

From the above wave diagrams, we can see that chaotic attractors from numerical simulation results and circuit simulation results are the same, which indicate that the design of controller is effective and feasible.

5. Conclusion

This paper presents a new fractional hyper-chaotic system, analyses its chaotic dynamics properties. Both experiments of numerical calculation and the circuit simulation to make sure the existence of chaotic attractors, while having hyper-chaotic dynamics characteristics. The new fractional order hyper-chaotic system has super chaotic phenomena, and a broad research and application value because of a wide ranges $q$ value, in $q = 0.80 - 0.99$ (step is 0.01). Finally designs a simple and effective linear feedback controller, experimental results also validate the effectiveness of the controller, and the controller has a universality, we can apply it to other fractional order control of hyperchaotic system.

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