An Improved Particle Filter with Applications in Ballistic Target Tracking

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Abstract: In this paper, we present an improved particle filter algorithm for ballistic target tracking, the quadrature Kalman particle filter (QKPF). The proposed algorithm uses quadrature Kalman filter (QKF) for generating the proposal distribution. The QKF is a recursive, nonlinear filtering algorithm developed in the Kalman filtering framework. It linearizes the nonlinear functions using statistical linear regression method through a set of Gaussian-Hermite quadrature points, need not to compute the Jacobian matrix and is easy to implement. Moreover, the generated proposal distribution integrates the latest measurements into system state transition density, so it can match the posterior density well. The simulation results show that QKPF is a viable alternative for sequential estimation in nonlinear dynamic models.

Keywords: Ballistic target tracking, Quadrature Kalman filter, Importance density function, Particle filter.

1. Introduction

Ballistic target tracking using conventional radar measurements is a difficult problem due to the inherent nonlinearity in both the target motion model and the radar observation model. The main goal is to estimate in a recursive fashion the unknown kinematic state, e.g. position and velocity, of the target given a sequence of noisy position measurements. In general, a closed-form analytical expression for the minimum mean square error (MMSE) estimate of the hidden state cannot be obtained in the ballistic target tracking problem due to the nonlinearity. We resort then to a sequential importance sampling method, referred to in the literature as particle filtering (PF) [1], to approximate the optimal MMSE estimate.

Farina et al. proposed in [2] the application of a bootstrap particle filter (PF) for the problem of ballistic target tracking assuming a nonlinear target motion model and a linear observation model, with both models specified in Cartesian coordinates. In this paper, we use the same motion model as in [2], but assume an alternative nonlinear observation model in polar coordinates. Furthermore, instead of using the standard bootstrap filter, we present an improved particle filter to this ballistic target tracking problem. It is known that the PF require the design of proposal distribution that can approximate the posterior distribution reasonably well. The standard particle filter is to sample from the probabilistic model of the states evolution (transition prior). This strategy can, however, fail if the new measurements appear in the tail of the prior or if the likelihood is too peaked in comparison to the prior. To overcome this problem, several techniques based on linearization have been proposed in the literature. For example, the extended Kalman particle filter
(EKF) [2] and the unscented particle filter (UPF) [3, 4], this two algorithms use EKF [3] and the unscented filter (UF) [5] to generate the proposal distribution respectively. Recently, in [6], the quadrature Kalman filter (QKF) algorithm is proposed. The QKF linearizes the nonlinear functions using statistical linear regression method through a set of Gaussian-Hermite quadrature points. It has been proved that QKF has higher estimation accuracy than the EKF and the UF [6]. In this paper, we use QKF to generate the proposal distribution and obtain a new kind particle filter, i.e. the quadrature Kalman particle filter (QKPF).

This paper is divided into 5 sections. Section 1 is this introduction. In section 2, we review briefly the target motion and the radar measurement models. In section 3, we present the proposed particle filter tracker. In section 4, we discuss the performance of our algorithm. Finally, section 5 summarizes the main results in this paper.

2. The Model

The plant selected in this paper is from literature [1, 3] and the geometry relation is illustrated in Fig. 1. It is assumed that the body is falling vertically, the effect of gravity is negligible.

\[ \dot{x}_i(t) = \omega_i(t), \]

where \( \omega_1(t) \), \( \omega_2(t) \), and \( \omega_3(t) \) are the zero-mean, uncorrelated noises with covariances given by \( Q(t) \) and \( \gamma \) is the constant \((5 \times 10^{-7})\) that relates the air density with altitude. The range at time \( t \), \( z(t) \), is

\[ z(t) = \sqrt{(M^2 + [x_1(t) - H]^2)} + r(t), \]

where \( r(t) \) is the uncorrelated observation noise with covariance \( R(t) = 10^5 \text{ ft}^2 \). The measurements are made with a frequency of 1 Hz.

The initial true state of the system is \( \dot{x}(0) = [3 \times 10^5 \quad 2 \times 10^4 \quad 10^{-5}] \), and the initial estimates and covariance matrixes of these states are

\[ \dot{x}(0 | 0) = [3 \times 10^5 \quad 2 \times 10^4 \quad 10^{-5}] \]

\[ P(0 | 0) = \begin{bmatrix} 10^6 & 0 & 0 \\ 0 & 4 \times 10^6 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \]

In this experiment, we don’t introduce any process noise into the simulation, \( Q(k) = 0 \), for both filters.

3. Quadrature Kalman Particle Filter

3.1. Particle Filter

The PF is a computer-based method for implementing an optimal recursive Bayesian filter by Monte Carlo simulations. The central idea is to represent the required probability density function (PDF) by a set of random samples (particles). As the number of particles is increased, the representation of the required PDF becomes more accurate.

The PF can be described as follows. Assume that we have a set of random samples (particles) \( \{x_{i-1}^{(k)}, i = 1, \ldots, N\} \) from the posterior density, \( p(x_{k-1} | z_{1:k-1}) \), at time \( k - 1 \). The PF is an algorithm for propagating and updating the set of random samples \( \{x_{i-1}^{(k)}, i = 1, \ldots, N\} \) to a new set of random samples at time \( k \), \( \{x_{i}^{(k)}, i = 1, \ldots, N\} \), which are approximately distributed as the posterior density \( p(x_{k} | z_{1:k}) \). Denote \( x_{0:k} = \{x_0, x_1, x_2, \ldots, x_k\} \) and \( z_{0:k} = \{z_0, z_1, \ldots, z_k\} \) be the states and observations up to time step \( k \), respectively.

In PF algorithm, the choice of proposal distribution is the key problem. In this paper, we use QKF algorithm to generate proposal distribution. And
next, we first describe the QKF algorithm, and then give the detailed QKPF algorithm.

3.2. Quadrature Kalman Filter

The QKF algorithm is developed from the statistical linear regression view rather than the numerical integration perspective. The QKF linearizes the nonlinear functions using statistical linear regression method through a set of Gaussian-Hermite quadrature points that parameterize the Gaussian density. It has been proved that QKF has higher estimation accuracy than the EKF and the UF [6].

Algorithm 1: Quadrature Kalman Filter

- Time update:
  1) Assumed that at time $k$ the posterior density function $p(x_{k-1} | z_{k-1}) = \mathcal{N}(x_{k-1}^\ast, \tilde{x}_{k-1}^\ast, P_{k-1}^x)$ is known, factorize
    $$P_{x|k-1} = S_{k-1, k-1}^x S_{k-1, k-1}^\top,$$
    (5)
  2) Compute the quadrature points $\{X_{j, k-1}\}_{j=1}^m$ as:
    $$X_{j, k-1} = S_{k-1, k-1}^x \tilde{x}_j + \tilde{x}_{k-1}^\ast,$$
    (6)
  3) Evaluate the propagated quadrature points $\{X_{j, k|k-1}\}_{j=1}^m$:
    $$X_{j, k|k-1} = f(X_{j, k-1}), \quad j=1, \ldots, m,$$
    (7)
  4) Estimate the predicted state and the corresponding error covariance:
    $$\hat{x}_{k|k-1} = \sum_{j=1}^m \omega_j X_{j, k|k-1},$$
    (8)
    $$P_{k|k-1} = Q_k + \sum_{j=1}^m \omega_j (X_{j, k|k-1} - \hat{x}_{k|k-1})(X_{j, k|k-1} - \hat{x}_{k|k-1})^\top,$$
    (9)
    where $Q_k$ is the process noise covariance.

- Measurement update step:
  1) Factorize
    $$P_{z|k} = S_{z|k}^x S_{z|k}^\top,$$
    (10)
  2) Evaluate the quadrature points $\{X_{j, k|k}\}_{j=1}^m$:
    $$X_{j, k|k} = S_{z|k}^x \tilde{z}_j + \tilde{x}_{k|k},$$
    (11)
  3) Evaluate the propagated quadrature points $\{Z_{j, k|k-1}\}_{j=1}^m$ as:
    $$Z_{j, k|k-1} = h(X_{j, k|k-1}), \quad j=1, \ldots, m,$$
    (12)
  4) Estimate the predicted measurement:
    $$\hat{z}_{k|k-1} = \sum_{j=1}^m \omega_j Z_{j, k|k-1},$$
    (13)
  5) Estimate the innovation covariance matrix:
    $$P_{z|k} = \sum_{j=1}^m \omega_j (Z_{j, k|k-1} - \hat{z}_{k|k-1})(Z_{j, k|k-1} - \hat{z}_{k|k-1})^\top,$$
    (14)
    where $R_k$ is the measurement noise covariance.
  6) Estimate the cross covariance matrix:
    $$P_{x|k} = \sum_{j=1}^m \omega_j (X_{j, k|k-1} - \hat{x}_{k|k-1})(Z_{j, k|k-1} - \hat{z}_{k|k-1})^\top,$$
    (15)
  7) Estimate the Kalman gain
    $$K_k = P_{z|k} S_{z|k}^x K_k^\top,$$
    (16)
  8) Estimate the updated state and the corresponding error covariance:
    $$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - \hat{z}_{k|k-1}),$$
    (17)
    $$P_{k|k} = P_{k|k-1} - K_k P_{z|k} K_k^\top,$$
    (18)
  9) Factorize the $P_{z|k}$
    $$P_{z|k} = S_{z|k}^x S_{z|k}^\top,$$
    (19)

At the end of the measurement update, we have the posterior density $p(x_k | z_k) = \mathcal{N}(\hat{x}_{k|k}, \hat{x}_{k|k}, P_{k|k})$.

About quadrature points $\tilde{x}_j$ and the corresponding weights $\omega_j$, we give the following computation method [6].

Consider a scalar random variable $x$ having a Gaussian probability density $\mathcal{N}(x; 0, 1)$. The expected value of the function $g(x)$ can be approximated as

$$E(g(x)) = \int g(x) \mathcal{N}(x; 0, 1) dx = \sum_{j=1}^m \omega_j g(\tilde{x}_j),$$

(20)

Instead of finding quadrature points using root-finding methods, which may be mathematically unstable, a computationally better approach is presented in [1] to find the quadrature points and
weights. This approach exploits the relationship between orthogonal polynomials and tri-diagonals.

Suppose $J$ is a symmetric tridiagonal matrix with zero diagonal elements and

$$J_{j,j+1} = \sqrt{j/j+2}, \quad 1 \leq j \leq (m-1), \tag{21}$$

where $m$ is the quadrature point number, in this paper, we take 3 quadrature point. Then 3

$$x_{n,1}, x_{n,2}, (x_{n,1})^{\top}, \tag{21}$$

where $m$ is the quadrature points number, and $\xi_j$ is the $j$-th eigenvalue of $J$; and the corresponding weight $\omega_j = \left(\eta_j\right)^2$, where $(\eta_j)_i$ is the first element of the $j$-th normalized eigenvector of $J$.

3.3. Quadrature Kalman Particle Filter (QKPF)

In this paper, the new particle filter, QKPF, use the QKF to generate the proposal distribution. In section 3.2, we have given the detailed QKF algorithm, the QKPF algorithm is now given by algorithm 2.

Algorithm 2: The Quadrature Kalman Particle Filter

- Initialization: at time $k = 0$
  1. For $i = 1, \ldots, N$
     Draw the particles $x_0^{(i)}$ from the prior $p(x_0)$ and set,
     $$x_0^{(i)} = E[x_0^{(i)}],$$
     $$P_0^{(i)} = E[(x_0^{(i)} - x_0^{(i)})^2],$$
     $$P_0^{(i)} = S_0^{(i)}(S_0^{(i)})^T$$
  2. For each $i = 1, \ldots, N$
     The initial important weight $w_0^{(i)} = 1/N$.
     • Prediction and update: For each time $k \geq 1$
       1. For $i = 1, \ldots, N$, update the particles with the QKF
          1) Calculate the quadrature points $X_{j,...,m,k-1}^{(i)}$.
             $$X_{j,...,m,k-1}^{(i)} = S_{k-1}^{(i)} \xi_j + \tilde{x}_{k-1}^{(i)}, j = 1, \ldots, m,$$
          2) Time update: Propagate particle into future
             $$X_{j,...,m,k-1}^{(i)} = f(X_{j,...,m,k-1}^{(i)}),$$
             $$\tilde{x}_{k-1}^{(i)} = \sum_{j=1}^m \omega_j X_{j,...,m,k-1}^{(i)}.$$
       2. For each $i = 1, \ldots, N$
          The initial important weight $w_0^{(i)} = 1/N$.
          • Prediction and update: For each time $k \geq 1$
            1. For $i = 1, \ldots, N$, update the particles with the QKF
               1) Calculate the quadrature points $X_{j,...,m,k-1}^{(i)}$.
                  $$X_{j,...,m,k-1}^{(i)} = S_{k-1}^{(i)} \xi_j + \tilde{x}_{k-1}^{(i)}, j = 1, \ldots, m,$$
               2) Time update: Propagate particle into future
                  $$X_{j,...,m,k-1}^{(i)} = f(X_{j,...,m,k-1}^{(i)}),$$
                  $$\tilde{x}_{k-1}^{(i)} = \sum_{j=1}^m \omega_j X_{j,...,m,k-1}^{(i)}.$$
   3) Measurement update: Incorporate new observation
      $P_k^{(i)} = Q_k + \sum_{j=1}^m \omega_j (X_{j,...,m,k-1}^{(i)} - \tilde{x}_{k-1}^{(i)}) \times (X_{j,...,m,k-1}^{(i)} - \tilde{x}_{k-1}^{(i)})^T$

   3. For $i = 1, \ldots, N$, evaluate the normalized weights
      $w_k^{(i)} = w_{k-1}^{(i)} \frac{p(z_k | x_k^{(i)})}{q(z_k | x_{0:k-1}^{(i)}, z_{k-1}^{(i)})}$
      $$w_k = \sum_{i=1}^N w_k^{(i)}, \quad \tilde{w}_k^{(i)} = w_k^{(i)}/w_k$$

where $p(z_k | x_k^{(i)})$ is the likelihood function of the measurement $z_k$.

4. Resample:
   If $N_{\text{eff}} < N_{\text{th}}$ ( $N_{\text{eff}} = \frac{1}{\sum_{k=1}^N (w_k^{(i)})^2}$ is sample volume, $N_{\text{th}}$ is a some given threshold value), set
   $\tilde{w}_k^{(i)} = 1/N$, and resample with $\{x_k^{(i)}, \tilde{w}_k^{(i)}, i = 1, \ldots, N\}$.

• Output: the experience probability distribution of filtering distribution, system state estimate and covariance are given respectively:
   $$\hat{p}(x_k | z_{1:k}) = \frac{1}{N} \sum_{i=1}^N \delta(x_k - x_k^{(i)}),$$
\[
\hat{x}_k = \frac{1}{N} \sum_{i=1}^{N} x_k^{(i)},
\]

\[
P_k = \frac{1}{N} \sum_{i=1}^{N} (\hat{x}_k - x_k^{(i)}) (\hat{x}_k - x_k^{(i)})^T
\]

Note: The initial quadrature points and the associated weights, \(\{\xi_j, \omega_j\}_{j=1}^n\), are determined off-line in advance.

### 4. Simulation Results

The simulated target is tracked over 60 time steps. In the experiment, the number of the particles is \(N = 200\). Furthermore, about the QKPF algorithm, the higher the number \(m\) of quadrature points (per axis) is, the lesser the estimation error becomes. But, the estimation accuracy comes with an increased computational complexity. In this paper, we choose the number \(m = 3\). If we choose the numbers \(m = 5\), the estimation accuracy will be higher.

The following figures give the position, velocity and ballistic coefficient estimation error respectively. From the Fig. 2 to Fig. 4, we can see that the estimation errors of QKPF algorithm are all lower than the PF. So, the simulation experiment improve that the estimation accuracy of QKPF is higher than that of the PF apparently.

![Fig. 2. Position estimation error.](image)

![Fig. 3. Velocity estimation error.](image)

![Fig. 4. Ballistic coefficient estimation error.](image)

### 5. Conclusion

We presented in this paper a quadrature Kalman particle filter for ballistic target tracking problem. The proposed filter adopts the quadrature Kalman filter to generate the proposal distribution. Compared with the PF algorithm, the QKPF algorithm improved the numerical stability and the numerical accuracy, but at the expense of increased computational complexity. we conclude that the QKPF algorithm is a good alternative for the ballistic target tracking problem being studied.

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### References


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