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Approximation Modeling of Minimal Curved Surface and its Optimization Algorithm Based on Linear Partial Differential Equation

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Abstract: With the wide application of computers in various industries, Computer Aided Design (CAD) has become the measurement criteria for the development degree of national modernization of science and technology, as well as industrial modernization. CAD needs to perform information processing to whole process of product design, including geometric modeling, engineering drawing, calculation and analysis, etc. Among them, the geometric modeling design is the core technology of the CAD, which provides basic data for product design. This paper performs research on approximation modeling of minimal curved surface based on the linear partial differential equation of CAD, and optimizes its algorithm. *Copyright © 2014 IFSA Publishing, S. L.*

Keywords: CAD, Linear partial differential equations, Approximation modeling of minimal curved surface, Algorithm, Research.

1. Introduction

With the rapid development of computer application, the technology of computer aided geometric design emerges new development trend. In the field of engineering, due to the actual demand, CAD often needs to start drawing curved surface which has special geometrical properties. Therefore, in recent years, more and more scholars begin to pay close attention to the curved surface modeling technology of special geometric properties. What is the most high-profile are the developable surface and the minimum surface, developable surface is comparatively mature in CAD application, in the field of differential geometry, the minimal surface performs system study through a lot of work, but it is just at the beginning stage.

Minimal surface is closely related to partial differential equations. In fact, minimal surface is a special kind of nonlinear partial differential curved surface, the purpose is to simplify the calculation and construction process, performing approximation modeling of minimal surface with linear partial differential curved surface is a reasonable choice. But due to the shape of the linear partial differential curved surface is completely determined by the boundary conditions, so it has the advantages of less shape parameter and low mathematical background requirements to users. But it is incompatible with the traditional CAD modeling, and it has less degree of freedom, all the shape parameters are global parameters, it is not convenient for local control and confronts more limitations in practice. In order to overcome the above shortcomings, this paper studies

the B-spline curve approximation of linear partial differential curved surface through adopting the traditional node configuration method and the finite element method, but because the linear partial differential curved surface is a special case, using this algorithm to achieve the minimal surface approximation will make it have very big error, so based on the idea of optimization, the new algorithm is given and optimized again with the segment properties of linear partial differential curved surface. This paper compares this algorithm with the B-spline curve approximation algorithm of traditional linear partial differential curved surface on the approximation precision, by which to prove the optimization of algorithm.

2. Harmonic B - B Curved Surface on Angle Domain

Suppose there are (n+1)(n+2)/2 point vector Ti, j, k∈ 3,i,j $\Re$,k≥ 0, i + $j + k = n$, thus call

$$X(u,v,w) = \sum_{i+j+k=n} B^n_{i,j,k}(u,v,w) T_{i,j,k}(u,v,w) \in T, u,v,w \geq 0,$$
$$u + v + w = 1$$

to be n Bernstein-Bézier parametric curved surface on triangular domain T, for short, it is B-B curved space, here

$$B^N_{i,j,k}(u,v,w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k$$

is called n Bernstein odd function, Ti, j, refers to as fixed control point of curved surface .For convenient statement below, denote(n-k+2) fixed control point $\left\{T_{i,j,k-1}\right\}_{i+j=n-k+1}$ as the control vertex on the kth level of this curved surface.

According to the property of primary function $B^n_{i,j,k}(u,v,w)$, we could get the following derivation formula.

$$\frac{\partial X(u,v,w)}{\partial u} = n \sum_{i+j+k=n-1} B^{n-1}_{i,j,k}(u,v,w)(T_{i+1,j,k} - T_{i,j,k+1})$$
$$\frac{\partial X(u,v,w)}{\partial v} = n \sum_{i+j+k=n-1} B^{n-1}_{i,j,k}(u,v,w)(T_{i,j+1,k} - T_{i,j,k+1})$$
$$\frac{\partial X(u,v,w)}{\partial w} = n \sum_{i+j+k=n-1} B^{n-1}_{i,j,k}(u,v,w)(T_{i,j,k+1} - T_{i+1,j,k})$$

That the B - B curved surface on triangular domain is harmonic surface refers to that it satisfies the following harmonic equation:

$$X_{uu} + X_{vv} = 0$$

Due to harmonic functions still remains to be harmonic function after orthogonal transformation, so in the case that domain of definition is a non right triangle (such as the marker diagram Fig. 1(a)), it needs to solve second derivative for curved surface function on the direction of u parameter and perpendicular to the u parameters respectively (such as the mark diagram Fig. 1(b)), namely the operator

$$\Delta = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)$$

should be transformed to be

$$\Delta^{\perp} = \left[\frac{\partial^2}{\partial u^2} + \left(\frac{\partial^2}{\partial u^2} \right)^{\perp} \right].$$

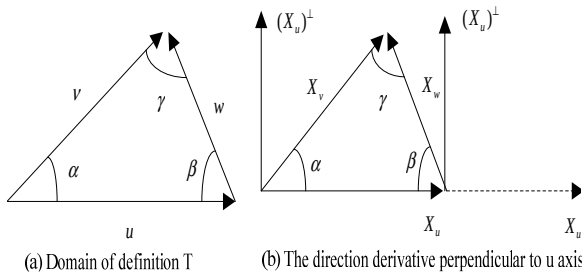


Fig. 1. The used notation sketch.

3. The Minimal Curved Surface Approximation Based on Linear Partial Differential Equation

In the modeling of linear partial differential curved surface, the order of partial differential equations is generally up to six orders at most and is even, so partial differential operator is first defined.

$$L = L\left(\frac{\partial^2}{\partial u^2}, \frac{\partial^2}{\partial v^2}, \frac{\partial^4}{\partial u^4}, \frac{\partial^2}{\partial u^2 \partial v^2}, \frac{\partial^4}{\partial v^4}, \frac{\partial^6}{\partial v^6}, \frac{\partial^6}{\partial u^4 \partial v^2}, \frac{\partial^6}{\partial u^2 \partial v^4}, \frac{\partial^6}{\partial v^6}\right)$$

In this paper, the main results are the PDE curved surface X (u, v) that obtained through approaching k order partial differential equation in the following forms with Bézier curved surface,

$$\begin{cases} LX(u,v) = F(u,v) \\ \frac{\partial^i X(0,v)}{\partial u^i} = f_i(v), \frac{\partial^i X(1,u)}{\partial u^i} = g_i(v) \\ \frac{\partial^j X(u,0)}{\partial v^j} = r_j(u), \frac{\partial^j X(u,1)}{\partial v^j} = g_j(u), i, j = 0, \ldots\ldots, \frac{k}{2} - 1 \end{cases},$$

where F(u, v) is binary algebraic polynomials.

3.1. The Approximation Algorithm of Minimal Curved Surface Based on Linear Partial Differential Equation

In curved surface modeling of partial differential equation, the boundary curve generally is circular curve or polynomial curve. But circular curve can't be denoted by minimal curve precisely. So in the case of boundary curve is circular curve, it needs to construct minimal curve to approximate this boundary curve. Performing minimal curve fitting to borderline curve through using least square method could be used in the case of boundary curve is other non-polynomial curve.

To make it easier to see, we let $\deg^u(F(u,v))$ denote the highest times of u in F(u,v) and let $\deg^u(F(u,v))$ denote the highest times of v in F(u,v), $\deg(f_0(v))$, $\deg(g_0(v))$, $\deg(r_0(u))$, $\deg(s_0(u))$, denote the times of boundary curved surface $f_0(v), g_0(v), r_0(u)$ and $s_0(u)$ respectively.

3.2. Algorithm: Approximation Algorithm of Minimal Curved Surface in Partial Differential Equation

1) Order.

$$n = \max\{k+1, \deg(r_0(u)), \deg(s_0(u)), \deg^u(F(u,v))\}$$
$$m = \max\{\deg(f_0(v)), \deg(g_0(v)), \deg^v(F(u,v))\}$$

Let approximation solution of minimal surface form be

$$P(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_i^n(u) B_j^m(v) a_{ij} ,$$

where a_{ij} is the unknown control vertice.

2) Through bringing P(u,v) into preceding approximate solution, we obtain complementary function R(u,v)=LP(u,v)-F(u,v), where

$$\deg^u(R(u,v)) = \max\{n-k, \deg^u(F(u,v))\}$$

$$\deg^u(R(u,v)) = \max\{m-k, \deg^v(F(u,v))\}.$$

Order $l = \deg^u(R(u,v)), r = \deg^v(R(u,v))$, so R(u,v) could be denoted as:

$$R(u,v) = \sum_{i=0}^{l}\sum_{j=0}^{r} B_i^l(u) B_j^r(v) b_{ij}$$

Obviously, b_{ij} is linear function of a_{ij}. Because it is obtained by the ascending order and derivation formula of minimal surface.

3) Construct objective function

$$M = \sum_{i=0}^{l}\sum_{j=0}^{r} b_{ij}^2(a_{00}, a_{01}, \ldots, a_{mn}),$$

solve the following constrained optimization problem.

$$\min M = \sum_{i=0}^{l}\sum_{j=0}^{r} b_{ij}^2(a_{00}, a_{01}, \ldots, a_{mn})$$

so as to make

$$\frac{\partial^i P(0,v)}{\partial u^i} = f_i(v), \frac{\partial^i P(1,v)}{\partial u^i} = g_i(v),$$
$$\frac{\partial^j P(u,0)}{\partial v^j} = r_j(u), \frac{\partial^j P(u,1)}{\partial v^j} = s_j(u), i, j = 0, \ldots, \frac{k}{2} - 1.$$

Withdraw min M into solution problem of linear equations set through using Lagrange multiplication method, thus we can get the answer of a_{ij}.

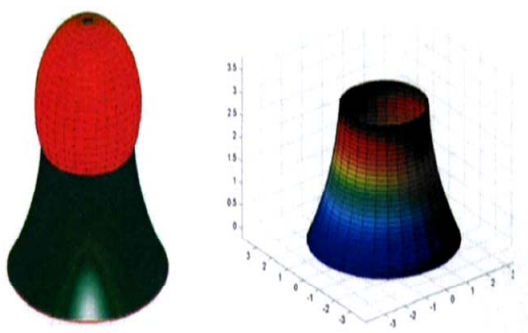
4) Bring a_{ij} into solving the equation

$$P(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_i^n(u) B_j^m(v) a_{ij}$$

We could calculate the approximation value P(u,v) of minimal surface X(u,v).

Fig. 2 and Fig. 3 are two calculating examples of using the algorithm respectively, the minimal surface is the transition surface of circle and one sphere. It is the solution of following boundary value problem:

$$\begin{cases} (\frac{\partial^2}{\partial u^2} + a\frac{\partial^2}{\partial v^2}) X(u,v) = 0 \\ x(0,v) = \sqrt{r^2 - (H - z_0)^2}\cos v, y(0,v) = \\ \quad = \sqrt{r^2 - (H - z_0)}\sin v, z(0,v) = H, \\ x(1,v) = R\cos v, y(1,v) = R\sin v, z(1,v) = 0 \end{cases}$$



(a) Exact solution (b) Approximation solution

Fig. 2. The exact solution and the minimal surface approximation of boundary value problem when a=1.

In Fig. 2(a), the green smooth surface is the exact solution of the boundary value problem, in which a=1, r-2, R=3, z0=4, H=3; Fig. 2(b) is the approximate solution of double five minimal surface; when the green smooth surface is a = 3. Fig. 3(a) is the exact solution; Fig. 3(b) is approximation surface.

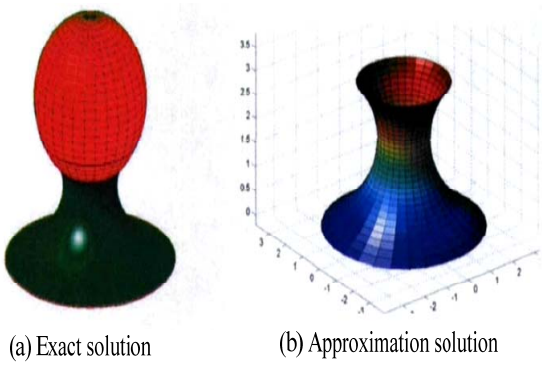


Fig. 3. The exact solution and the minimal surface approximation of boundary value +problem when a=3.

3.3. Optimization Algorithm

We could see from

$$\min M = \sum_{i=0}^{l}\sum_{j=0}^{r} b_{ij}^2(a_{00}, a_{01}, \ldots, a_{mn})$$

that objective function M is actually the Euclidean norm of control vertex on complementary function R (u, v), or you can use the convex hull of minimal surface to get the upper bound of the R (u, v):

$$R(u,v) \le \max_i \left\{ \max_j \left\{ \left| b_{ij} \right| \right\} \right\},$$
$$u, v \in \left[0,1\right]$$

So we can measure the approximation precision of the algorithm solution through above indicators, if above two indexes are not small enough, then we can use the segment properties of minimal surface to improve the approximation accuracy of solution.

For simple calculation, we divide region [0,1]*[0,1 into n*m sub-domain $[c_{i-1},c_i]*[d_j-1,d_j]$, where c_i=i/n, d_j=j/m, i=0,1.....,n, j=0,1...,m. Order the minimal surface defined on region $[c_{i-1},c_i]*[d_j-1,d_j]$ to be:

$$P_{ij}(u,v) = \sum_{h=0}^{n}\sum_{s=0}^{m} B_h^n\left(\frac{u-c_{i-1}}{c_i - c_{i-1}}\right) B_s^m \left(\frac{v-d_{j-1}}{d_j - d_{j-1}}\right) a_{hs}^{ij},$$
$$u \in \left[c_{i-1}, c_i\right], u \in \left[d_{j-1}, d_j\right]$$

Similarly, bring $P_{ij}(u,v)$ into

$$\begin{cases} LX(u,v) = F(u,v) \\ \frac{\partial^i X(0,v)}{\partial u^i} = f_i(v), \frac{\partial^i X(1,u)}{\partial u^i} = g_i(v) \\ \frac{\partial^j X(u,0)}{\partial v^j} = r_j(u), \frac{\partial^j X(u,1)}{\partial v^j} = g_j(u), i, j = 0, \ldots\ldots, \frac{k}{2} - 1 \end{cases}$$

We can get complementary function.

$$R_{ij}(u,v) = LP_{ij}(u,v) - F(u,v)$$
$$= \sum_{h=0}^{l}\sum_{s=0}^{r} B_h^l\left(\frac{u-c_{i-1}}{c_i - c_{i-1}}\right) B_s^r\left(\frac{v-d_{j-1}}{d_j - d_{j-1}}\right) b_{hs}^{ij},$$

where

$$l = \deg^u(R_{ij}(u,v)), r = \deg^v(R_{ij}(u,v)),$$

construct the objective function as below:

$$F(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{m}(c_i - c_{i-1}) \times (d_j - d_{j-1}) \sum_{h=0}^{l}\sum_{s=0}^{r}(b_{hs}^{ij})^2$$

At this time, the boundary conditions that approximation solution satisfies should be changed to

$$\frac{\partial^i P_{1l}(0,V)}{\partial u^i} = f_i(v), \frac{\partial^i P_{nl}(1,V)}{\partial u^i} = g_i(v),$$
$$\frac{\partial^j P_{s1}(u,0)}{\partial v^j} = r_j(u), \frac{\partial^j P_{sn}(u,1)}{\partial v^j} = s_j(u),$$
$$i, j = 0, \ldots, \frac{k}{2} - 1, l = 0,1,\ldots,n, s = 0,1,\ldots,m$$

In order to make approximation surface look more smooth, in addition to the boundary conditions, adjacent minimal surface should also satisfy continuity condition of C^r, namely r=0,1,2,.....,k, in other words

$$b_{ns}^{ij} = b_{0s}^{i+1,j},$$
$$\Delta^P b_{n-p,s}^{ij} = \Delta^P b_{0,s}^{i+1,j}, s = 0,\ldots,m, p = 0,\ldots,k$$
$$b_{hm}^{ij} = b_{h0}^{i,j+1}$$
$$\tilde{\Delta}^P b_{h,m-p}^{ij} = \tilde{\Delta}^P b_{h0}^{i,j+1}, h = 0,\ldots,n, p = 0,\ldots,k$$

Acquire a_{hs}^{ij} which makes objective function F (u,v) minimum under the constraints of above continuity condition and boundary conditions. This is a constrained optimization, as it is applied to the original algorithm, approximation solutions will be obtained.

The green surface in Fig. 4(a) is exact solutions of boundary value problems, and Fig. 4(b) is the approximation solution, such minimal surface is often used in the design of vase.

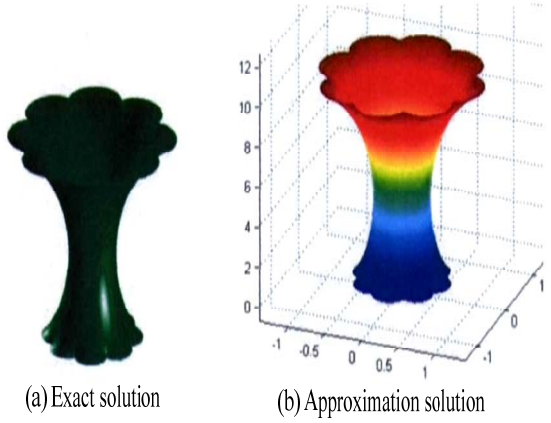


Fig. 4. Calculating examples of using optimization algorithm.

4. Summary

This paper mainly studies the approximation problem of minimal surface in linear partial differential equation.

First of all, we roughly know that the control mesh of harmonic B-B surface on any triangle domain is determined by the control vertice on the first and second layer. When the domain of definition is a non right triangle, there is no need for order elevation, thus relatively speaking, obtained results are fairly simple, the proof process does not need to convert partial differential equation into power base, which is more simple and understandable, and there is no need to solve system of linear equations, thus we could get the indication relationship between the control vertice on other layers and control vertice on the two layers at the bottom through iterative approach.

Second, we give the minimal approximation algorithm of partial differential equation surface, and optimize this algorithm through using the segment property of minimal surface. Compared with the traditional algorithm, there has been a qualitative change in the approximation precision, and it is easy for operation. The application scope of the algorithm is wider, which is equally applied properly to the high order partial differential equation, the non seven partial differential equation and the case that boundary curve is vector curve. So the algorithm not only has realized the compatibility between PDE modeling system and traditional CAD modeling system, but also provides a new method of minimal surface constraint model. It has promoted CAD modeling system to get progress in the field of construction project and design field.

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