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Fractional Fourier Transform Image Time-Frequency Information Research

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Abstract: Fractional Fourier Transform (FRFT) contains simultaneity the time-frequency information of the signal, and it is considered a new tool for time-frequency analysis. The energy distribution of image is analyzed and simulated in this paper, the characteristic of the amplitude and phase is extracted in Fractional Fourier domain of an image. The simulation results show that the FRFT can represent the time- frequency characteristic of an image, and the distribution changes with the variety of the fractional powers. Moreover, we also find some conclusion from the reconstructed image, and they are used the amplitude and phase information of FRFT. These conclusions will help to study how to use the FRFT in the area of image recognition and image edge extraction. *Copyright © 2014 IFSA Publishing, S. L.*

Keywords: Fractional Fourier transform, Time-frequency characteristics, Image processing, Feature extraction, Image reconstruction.

1. Introduction

Digital image processing technology is an emerging discipline to flourish, and is widely used in the digital photogrammetry, remote sensing image processing, bio-image processing, geographical information systems, and digital image processing technology transform domain has been a hot research field of image processing. The traditional two-dimensional discrete Fourier transform (DFT), a two-dimensional discrete cosine transform (DCT), a two-dimensional discrete sine transform (DST), and their transform ideas are that images are switched between the time-space domain and frequency domain, image information is extracted and image feature is analyzed. Two dimensional discrete cosine transform (DCT) with the highly energy accumulation has provide a theoretical basis for image compression and coding algorithms and still image and motion picture compression standard based on DCT transform has become a mainstream technology of image transmission and storage. However, it is well known that the above-mentioned transformation is the global transformation of time variable, and is unable to extract local spectral features. Gabor and short-time Fourier transform has an attempt to compensate for the lack of transformation above, but their window size does not change with frequency, and these restrictions can not solve the resolution problem of the time and the frequency. Therefore, the emerged wavelet transform has inherited and developed ideas of Gabor and short-time Fourier transform, as well as this has overcome that the window size does not change with frequency, the lack of a discrete orthogonal basis and other shortcomings. Wavelet transform time-frequency localization features make it ideally suited for image processing, and has been widely used in the image compression standard in recent years, image segmentation, image reconstruction technique, and become a hot research in image processing sector in recent years [1-4].

Fractional Fourier Transform (FRFT) is a new time-frequency analysis tool which is developed in recent years, and it is the Fourier transform of the generalized form. In essence, the signal makes the representation on the fractional Fourier domain, while this is the integration of signal information in the time domain and frequency domain. This new mathematical tool not only is closely linked with the Fourier transform, but also is very meaningful with other time-frequency analysis tools, and one has been widely used in optical system analysis, filter design, signal analysis, solving differential equations, phase recovery and pattern recognition field [5]. Most applied research of fractional Fourier transform in recent years is focused on the linear FM signal estimation, detection and filtering aspects, for its application is less in image processing. Image processing of fractional Fourier transform is only limited to the chirp digital watermark detection [6-8] of image. Therefore, to explore and analyze the image characteristics of the fractional Fourier domain, and to tap the image value of the fractional Fourier transform is of great significance.

Arrangements are as follows, Section 2 gives the definition of the fractional Fourier transform and its two-dimensional discrete algorithm; In part 3, First, to analyze and examine the image energy distribution characteristics in different FRFT domain from the simulation point of view, the energy accumulation of FRFT domain is assessed and analyzed by normalizing residual error factor; then the amplitude and phase of the image have included information by the simulation analysis of the FRFT dual-domain characteristics.

The results show that in any fractional Fourier domain, the image can reflect the space-frequency domain characteristics. The space-frequency domain characteristics of the image will change with the different order transformation. Relationship also reveals some important conclusions between the original image and image information which is reconstructed by the amplitude and phase of the images fractional Fourier transform. These conclusions are very meaningful for which the fractional Fourier transform is applied to image edge detection and recognition.

2. The fractional Fourier Transform (FRFT)

FRFT can be explained that the representation of the fractional Fourier domain is formed after the signal does counterclockwise rotation any angle around origin in the time-frequency plane axes, and this is a generalized form of Fourier transform.

FRFT of the signal $x(t)$ is defined as [9]

$$X_\alpha(u)=\left\{F^\alpha\left[x(t)\right]\right\}(u)=\int_{-\infty}^{\infty}x(t)K_\alpha(t,u)\,dt, \quad (1)$$

where FRFT transform kernel $K_\alpha(t,u)$

$$K_\alpha(t,u)=\begin{cases}\sqrt{\frac{1-j\cot\alpha}{2\pi}}\exp\left(j\frac{t^2+u^2}{2}\cot\alpha-tu\csc\alpha\right), & \alpha\neq n\pi \\ \delta(t-u), & \alpha=2n\pi \\ \delta(t+u), & \alpha=(2n\pm1)\pi\end{cases}, \quad (2)$$

The formula $\alpha = p\pi/2$ is the FRFT angle of rotation. For two-dimensional signal $x(s,t)$, the two-dimensional FRFT can be expressed as:

$$X_{p1,p2}(u,v)=F_{p2}^{t\to v}\left\{F_{p1}^{s\to u}\left[x(s,t)\right]\right\}, \quad (3)$$

By the discrete, FRFT can also be calculated by using digital methods. The most commonly used algorithm is the decomposition fast algorithm which is proposed by Ozaktas [10, 11]. The signals can be decomposed into FRFT convolution by the algorithm, the calculated results are compare similar with the continuous FRFT output. Decomposition type FRFT transformation matrix as follows:

$$F_p = DK_pJ, \quad (4)$$

where D and J are the twice inside difference between the matrix and the matrix of the extraction operation respectively, K_α is the discrete FRFT nuclear transforming matrix, namely:

$$K_p=\frac{A_\alpha}{2\Delta x}\exp\left(\frac{j\pi(\cot\alpha)m^2}{(2\Delta x)^2}-\frac{j2\pi(\csc\alpha)mn}{(2\Delta x)^2}+\frac{j2\pi(\cot\alpha)n^2}{(2\Delta x)^2}\right)|m|,|n|\leq N, \quad (5)$$

A discrete dimensionless normalized fractional Fourier transform is defined as:

$$X_p(u)=\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}\exp\left[j\pi\cot(\alpha)u^2\right]\times\int_{-\infty}^{+\infty}x(t)\exp\left[j\pi\cot(\alpha)t^2\right]\exp\left[-j2\pi\csc(\alpha)tu\right]dt, \quad (6)$$

3. The Discrete Fractional Fourier Transform Analysis in Image

3.1. Energy Distribution of the Image in Fractional Fourier Domain

The discrete transform energy distribution of a two-dimensional image reflects the characteristics of the transforming image. Never lossy compression point of view, the transform purpose is that the energy as much as possible is focused to a small number of several coefficients after the image is transformed. So by quantified, and only a few coefficients is not zero, which can get higher compression ratio. It is well known that DCT energy accumulation is superior to other transformation. The purpose of this section is the energy accumulation of the image FRFT domain. To this end, we first define a normalized residual error factor ρ. For the $M \times N$ image, the coefficient of the FRFT is $F^{\alpha}(k,h)$. According to the qualitative analysis, the FRFT domain energy is also concentrated in the central region [8], this definition:

$$\rho = \frac{\sum_{(k,h)\in r}\sum \left|F^{\alpha}(k,h)\right|^2}{\sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1}\sum_{h=-\frac{N}{2}}^{\frac{N}{2}-1}\left|F^{\alpha}(k,h)\right|^2}, \quad (7)$$

where the r corresponding area is $k=(0,\left|\frac{M}{4}-1\right|), h=(0,\left|\frac{N}{4}-1\right|)$.

In order to reflect the universality of the conclusion, the simulation selected seven images with different texture features and different sizes of standard gray-scale image ('Lena' (512 × 512), 'baboon' (512 × 512), 'bridge,' (256 × 256), 'cameraman' (256 × 256), 'rice' (256 × 256), 'moon' (358 × 536), 'saturn would' (438 × 328)).

In order to simplify the simulation calculation, the transform order is $p_1 = p_2$, and after changing the order of p, the changes of the normalized residual error factor ρ is investigated, the simulation results shown in Fig. 1 and Fig. 2.

From the simulation results, it can be seen that the energy of the image in the fractional Fourier domain shows a certain regularity:

1) With increasing the order of the transformation (due to the symmetry of the FRFT order, changes in the interval [0,1] have general), the image share energy of a quarter of the coefficient increases in the middle of the FRFT domain. We can see that the energy of the image in fractional Fourier domain shows the aggregation, and its energy distribution tends to be the center FRFT domain of two-dimensional coordinate plane. Transform order $p1 = p2 \le 0.5$, with the increase of the transformation order, the aggregation degree of the image energy is also significantly improved. When the angle change is near 0.7, the energy aggregation in this FRFT domain area has reached more than 90 %.

2) Since the fractional Fourier transform meets the Parseval criteria with the relationship of the energy conservation, and FRFT image also contains a time-frequency information, so the energy distribution in the space-frequency domain is also changing with change order. When the transform angle is the small $(p1 = p2 \le 0.5)$, the rotative angle which is corresponding to the time-frequency plane is less than $\pi/4$, the energy distribution in FRFT does not reflect well the aggregation, almost half of the energy are distributed in the airspace or scattered throughout the $N \times N$ region, the changing trend is relatively flat; when $(p1 = p2 > 0.5)$, the energy accumulation changes are more obvious with the increasing trend of the transform angle, the corresponding time-frequency plane rotation angle is close to $\pi/2$, the energy distribution is close to Fourier transform in the frequency domain, and the energy is concentrated in only a few coefficients. At that $p1 = p2 = 1.0$ time, FRFT degradation is on the Fourier transform, the image energy is to achieve the greatest degree of aggregation.

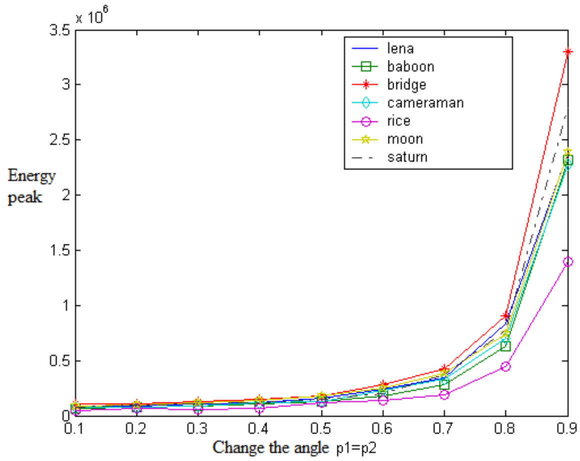

Fig. 1. The curve of the normalized residual error factor changes with the order number.

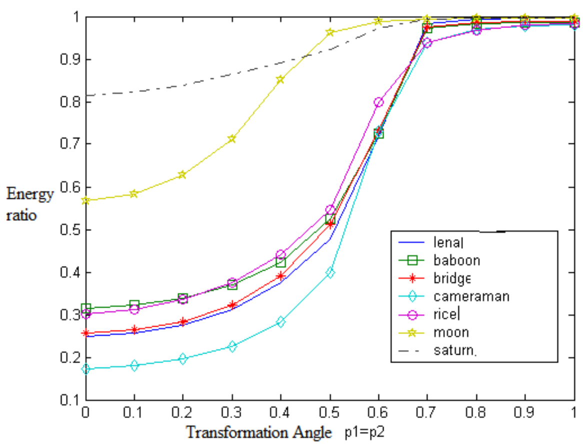

Fig. 2. FRF domain image energy peak curve with the change of the order.

From the above analysis, image FRFT has aggregation, but their aggregation depend the global on the Fourier transform, and the energy distribution with the order changing characteristics also reflects the characteristics of dual-domain information on FRFT.

3.2. Image Fractional Fourier Transform Amplitude and Phase Information

We know, for image processing and image receiver, the phase information is more important than amplitude information, and the phase information is the most vulnerable during transmission, the channel delay and Doppler shift will affect the phase information. It is necessary to study the phase and amplitude information and associated characteristics of the image by the FRF. In order to study the amplitude and phase characteristics of the image FRFT, the phase and amplitude characteristics are researched first by reviewing the traditional Fourier transform.

Assumption $F(k,h)$ is that the two-dimensional Fourier transform of the two-dimensional image $f(x,y)$

$$F(k,h) = FT_{2D} f(x,y), \tag{8}$$

$F(k,h)$ can be decomposed into the amplitude component and phase components, namely

$$F(k,h) = |F(k,h)| \bullet P(k,h) = A(k,h) \bullet P(k,h), \tag{9}$$

Among $A(k,h) = |F(k,h)|$ is amplitude function. $P(k,h) = F(k,h)/A(k,h)$ is the phase function. Respectively, $A(k,h)$ and $P(k,h)$'s two-dimensional Fourier inverse transform is to get $a(k,h)$ and $p(k,h)$. The Saturn image as an example simulation, the simulation results shown in Fig. 3.

The original image information is recovered from $a(k,h)$ in Fig. 3(b), which contains only the background information of the original image, its results are similar to the original image experienced a low-pass filter; the original image information is recovered from $p(k,h)$ in Fig. 3(c), which can be edge information.

And this shows that recovery image by the Fourier transform phase contains only the edge information of the original image, the result is similar to the original image through the high-pass filter, so that the FT can be used for image edge detection and pattern recognition[12, 13].

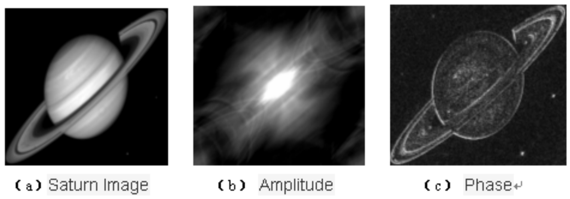

Fig. 3. The Saturn image and its FT magnitude part and the phase section.

Use similar methods to study the amplitude and phase characteristics of the image in fractional Fourier domain coefficient, $p1 = p2 = 0.01$, 0.1, 0.5, 0.8 were taken in the fractional Fourier domain, and the "Saturn" image two-dimensional fractional Fourier transform coefficients are divided into the amplitude function and phase function, respectively, by the Inverse transform in the fractional Fourier transform, the simulation results shown in Fig. 4 and Fig. 5.

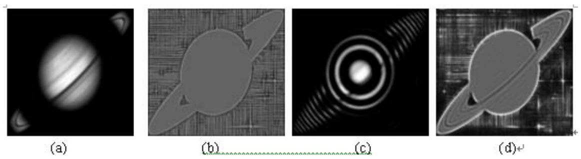

Fig. 4. p = 0.01 and p = 0.1, the Saturn images are recovered respectively by the FRFT magnitude function and FRFT phase function. (a) The image is recovered by FRFT magnitude function when p = 0.01. (b) The image is recovered by FRFT phase function when p = 0.01. (c) The image is recovered by FRFT magnitude function when p = 0.1. (d) The image is recovered by FRFT phase function when p = 0.1.

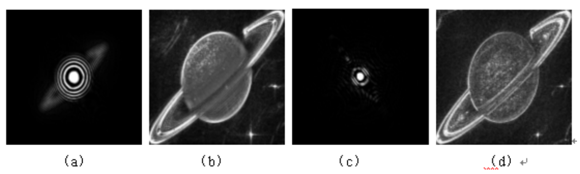

Fig. 5. p = 0.5 and p = 0.8, the Saturn images are recoved respectively by the FRFT magnitude function and FRFT phase function. (a) The image is recovered by FRFT magnitude function when p = 0.5. (b) The image is recovered by FRFT phase function when p = 0.5. (c) The image is recovered by FRFT magnitude function when p = 0.8. (d) The image is recovered by FRFT phase function when p = 0.8.

It is shown in Fig. 4 and Fig. 5 that the different p is taken, from the IFRFT images of the FRFT amplitude function, and the IFRFT images of the

FRFT phase functions, the following conclusions can be obtained:

a) When the transform order P is changed from small to big, the image is restored only by the phase function, the revealing edge of the original image has become increasingly clear, which is similar to the original image has gone through different cut-off frequency high pass filter. When p is small (0.01), it is correspond to the high-pass filter in the lower cutoff frequency, and low frequency components emerge, so that the extraction of edge is blur, as shown in Fig. 4(b); When p is large (0.8), it is correspond to the high-pass filter which cut-off frequency is higher, the majority of low-frequency components are filtered out, and the extracted edge is clearer, as is shown in Fig. 5(d), this time FRFT is degenerate into the basic FT.

b) Similarly, when the order P of the transform is changed from small to big, only by the amplitude function to restore the image, it is closer and closer to the original image background, which is similar to that the original image has experienced a low-pass filter in the different cut-off frequency. When p is small (0.01), it is correspond to the low-pass filter in the higher cutoff frequency, high-frequency component residue is more, but also the outline of the original image is clearly seen, as it is shown in Fig. 4(a); when p is large (for 0.8), it is correspond to the low-pass filter in the lower cutoff frequency, most of the high-frequency components is filtered out, the background of the original image is only show, as it is shown in Fig. 5(c).

c) In the transform, the order P is any other value, the restored image by FRFT phase function and amplitude function contains both the original image background and also includes the texture of the original image, as it is shown in Fig. 4(b), Fig. 4(c), Fig. 5(a), Fig. 5(b) below. In view of this situation, we can deduce that this is similar to that the original image has gone through the FRFT time-frequency filtering, and that is also about which the time-frequency plane is whirlled at an angle before filtering. If the cut-off frequency in frequency domain filter and bandwidth are fixed, when the rotation angle (order) is different, the projection is also different in the timeline and frequency axis, so the output frequency components are also different in the frequency domain filter. Performance is to restore the image, that is the phase function and the amplitude function have contained the frequency component which change with the order.

In short, when the order of the transformation is small, the i-recovery image by the amplitude function and phase function of the FRFT shows a strong image information, they reflects the strong airspace characteristics (Fig. 4); when the transform order number is gradually increasing, the image which is restored by the FRFT of amplitude function has contained in the original image airspace characteristics which is gradually weakened until it disappears, by the FRFT phase function to restore, the image has the edge texture features of the original image which is gradually increased, and it is compared according to Fig. 3, when the order increases in FRFT transform, the amplitude characteristics and phase one are closer and closer to one of the FT domain, and that is frequency domain features (Fig. 5). These conclusions reflect the two-domain characteristics of the time-frequency in FRFT domain.

4. Conclusions and Outlook

This paper is that only from the perspective of qualitative simulation, the image is studied in the fractional Fourier transform energy distribution, the amplitude and phase characteristics and association with the original image. Simulation results show that: the image energy accumulation in the Fractional Fourier transform is Related to the transformation order, its aggregation is strongly dependent on and is close to the extent of its Fourier transform; it is similar to the traditional Fourier transform, the FRFT phase function contains the image texture information, the edge information which is contained in the phase function are not the same with the changing angle, this may be that the fractional Fourier transform can be more flexible for image edge extraction and recognition. How the FRFT features are researched by a more comprehensive, in-depth study and the image time-frequency characteristics is extracted full by the advantage of its dual-domain representation, above is the author further study.

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