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ECG De-noising Based on Translation Invariant Wavelet Transform and Overlapping Group Shrinkage

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Abstract: Electrocardiogram (ECG) signal plays an important role in the diagnosis of cardiovascular disease. However, ECG signal is very faint and always affected by a variety of noise in the process of collecting. How to eliminate the noise effectively is an important issue and has been widely studied for many years. In this paper, we propose a new ECG de-noising method based on translation invariant (TI) wavelet transform and overlapping group shrinkage (OGS). The OGS is a new thresholding function, which is especially suitable for processing the large-amplitude coefficients form groups. The proposed method is tested on white Gaussian noise added the analog signals and ECG signals. Signal to Noise Ratio (SNR) and Root Mean Square Error (RMSE) are used to compare the performance of the proposed method with other de-noising methods. The experimental results indicate that the proposed de-noising method is the best in aspects of the improvement of SNR and remaining the geometrical characteristics of the ECG signals. *Copyright © 2014 IFSA Publishing, S. L.*

Keywords: ECG, Wavelet transform, Translation invariant, OGS, De-noising.

1. Introduction

ECG signal is one of the non-linear, non-stationary and weak biomedical signals, which can reflect human body heart activities and provide valuable information of the heart functional conditions. It is widely used in various kinds of heart disease diagnosis in clinic [1]. However, ECG signal is vulnerable to be corrupted by different noise during acquisition, such as baseline wander noise, power line interference, muscle contraction, motion artifacts etc. The noise will degrade the accuracy and precision of the analysis, so it must be eliminated in order to obtain a clean ECG signal for the accurate diagnosis of heart conditions.

Recently, many de-noising methods have been reported in literature for ECG noise reduction, most of which based on filter banks, adaptive filtering, independent component analysis (ICA), empirical mode decomposition (EMD) and wavelet de-noising techniques [2-4]. However, the methods of filters are not so effective when the signal components and the noise components are overlapping in the spectrum, it will remove not only the noise components but also the high frequency components of the non-stationary signal, which will cause further signal distortion.

Now with the development of wavelet theory, the application of signal processing method based on wavelet transform is more and more popular due to the advantages of the multi-resolution analysis and the better time-frequency analysis characteristics, it has been proved to be a powerful tool for non-stationary signal analysis. The wavelet thresholding de-noising method was first proposed by Donoho [5]. Since then, lots of de-noising methods based on the wavelet transform appeared for the non-stationary signals de-noising. Ying proposed a new threshold and shrinkage function based on the Multi-analysis wavelet threshold de-noising [6]. Han proposed a TI multiwavelet de-noising method for the ECG noise elimination [7].

In this paper, a new TI wavelet de-noising method with OGS algorithm is presented for the recovery of the signals contaminated by white additive Gaussian noise, which can suppress the Pseudo-Gibbs phenomenon in Q wave and R wave of the de-noised ECG signal. This paper is organized as follows: Section 2 introduces the wavelet transform, Section 3 introduces the OGS algorithm and section 4 explains the proposed de-noising method. Experimental results are discussed in Section 5. Finally, the conclusions are presented in Section 6.

2. Wavelet Transform

French scientists Morlet and Grossman put forward the concept of continuous wavelet transform when they made the analysis of seismic waves and found the traditional Fourier Transform can't meet the requirements of the local analysis in 1984 [8].

The wavelet transform describes a multi-resolution decomposition process which decomposes a signal into a set of wavelet basis functions, which play a key role in the multi-resolution analysis in wavelet domain. A "mother wavelet" is a small wave which has energy limited and concentrated. If the function $\psi(t)\in L^2(\mathbf{R})$ satisfies the property:

$$C_\psi = \int_{-\infty}^{+\infty} |\omega|^{-1} |\overline{\psi}(t)|^2 \, d\omega < \infty, \quad (1)$$

where $\overline{\psi}(t)$ is the Fourier Transform of $\psi(t)$. $\psi(t)$ is so called as a "mother wavelet", and after scaling or translation which can generate a family of continuous wavelet functions:

$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right) \qquad a,b\in \mathbf{R},\ a\neq 0, \quad (2)$$

where a is the scaling factor and b is the translation factor.

Thus for an arbitrary function $f(t)\in L^2(\mathbf{R})$, the expression for continuous wavelet transform is given as:

$$Wf(a,b) = \langle f, \psi_{a,b} \rangle = |a|^{-1/2} \int_{-\infty}^{+\infty} f(t)\psi^*\left(\frac{t-b}{a}\right)dt, \quad (3)$$

where $\psi^*(t)$ is the conjugate function of $\psi(t)$.

Computation of continuous wavelet coefficients at every possible scale is a fair amount of work and which generates an awful lot of data. In order to overcome this redundancy, we need a discretization processing. Discrete wavelet transform (DWT) can be obtained by discretization the scaling factor a and the translation factor b. In general, the selection of a subset of scales and positions are as given below:

$$a = a_0^m,\ b = nb_0a_0^m \quad a_0 > 1,\ b_0 \neq 0,\ m,n\in \mathbf{Z}, \quad (4)$$

The discrete wavelet functions can be expressed:

$$\psi_{m,n}(t) = a_0^{-m/2}\psi(a_0^{-m}t - b_0 n), \quad (5)$$

For the function $f(t)\in L^2(\mathbf{R})$, the DWT can be written by:

$$Wf_{m,n} = |a_0|^{-m/2} \int_{-\infty}^{+\infty} f(t)\psi^*(a_0^{-m}t - nb_0)dt \quad m,n\in \mathbf{Z}, \quad (6)$$

In particular, if $a_0 = 2$ and $b_0 = 1$, we can obtain the binary wavelet functions:

$$\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n), \quad (7)$$

The basic information of the signals will not loss in DWT, on the contrary, due to the orthogonality of the wavelet basis functions, the correlation between two points in wavelet space caused by redundancy will be eliminated, which makes the calculation error smaller. Therefore, DWT results in a more efficient and accurate analysis.

Due to the good ability of time-frequency localization characteristics, wavelet transform has been widely used in the field of signal de-noising [9]. The de-noising method is easy to be conducted by dealing with the wavelet coefficients to remove the unwanted noise and then reconstructed them. In general, it can realize the de-noising purpose of the noisy signals. However, due to the lack of translation invariance of the wavelet basis, it may produce the Pesudo-Gibbs phenomenon in the neighborhood of discontinuities. One method to suppress such artifacts, termed "cycle spinning", which first presented by Coifman and Donoho [10]. We can implement this method by shifting the noisy data, de-noising the shifted data, and then inverse-shifting the de-noised data. Repeating these steps for many times and averaging the several results to obtain the final reconstructed signal.

3. The OGS Algorithm

In recent years, many algorithms based on sparsity have been developed for signal de-noising. These algorithms often utilize the nonlinear scalar thresholding functions of various forms which have been devised so as to obtain sparse representations. For many natural signals (ECG signal or PCG signal), the variables of signals or coefficients are not only sparse but also exhibit a clustering or grouping property. For example, wavelet coefficients generally have inter and intra-scale clustering tendencies, and the large-amplitude values of the coefficients tend not to be isolated. However, ℓ_1-norm algorithm such as Basic Pursuit (BP) and other separable sparsity models such as Lasso algorithm do not capture the tendency of coefficients to group sparsity. Chen and Selesnick developed a simple translation-invariant thresholding algorithm which exploits the grouping properties of the signals or coefficients called overlapping group shrinkage (OGS) [11]. They used this algorithm for speech signal de-noising and good results were obtained. The principle of OGS algorithm can be described as follows:

Assuming the noisy observation $y(i)$ is defined:

$$y(i) = x(i) + \omega(i), \quad i \in I, \tag{8}$$

where $x(i)$ is the signal which has a group sparse property and $\omega(i)$ is the white Gaussian noise.

The purpose of de-noising is to estimate the "clean signal" from the noisy one. A generally an effective approach for deriving thresholding function is to formulate the following optimization problem:

$$\mathrm{x}^* = \arg\min_{\mathrm{x}} \left\{ F(\mathrm{x}) = \frac{1}{2}\|\mathrm{y}-\mathrm{x}\|_2^2 + \lambda R(\mathrm{x}) \right\}, \tag{9}$$

where $R(\mathrm{x})$ is the penalty function. If $R(\mathrm{x})$ is the separable form, such as (10), it significantly simplifies the task:

$$R(\mathrm{x}) = \sum_{i \in I} r(x(i)), \tag{10}$$

For example, if $R(\mathrm{x}) = |\,\mathrm{x}\,|$, then the solution to (9) is soft-thresholding corresponding to the MAP [12]:

$$\hat{\mathrm{x}} = \left(1 - \frac{\lambda}{\|\mathrm{y}\|}\right)_+ \mathrm{y}, \tag{11}$$

where $(x)_+ := \max(x, 0)$, the thresholding is λ. However, the OGS algorithm minimizes the cost function with the non-separable penalty function [11]:

$$R(\mathrm{x}) = \sum_{i \in I} \left[\sum_{j \in J} |x(i+j)|^2 \right]^{1/2}, \tag{12}$$

where $I = \{0, ..., N-1\}$, $J = \{0, ..., K-1\}$, the set J defines the group, the index i is the group index, and j is the coefficient index within group i. Each group has the same size of $|\,J\,|$. The OGS thresholding function can be derived as follows using the Majorization-minimization (MM) method [13]:

$$x^{(k+1)}(i) = \begin{cases} \dfrac{y(i)}{1+\lambda r(i; \mathrm{x}^{(k)})}, & i \in I' \\ 0 & , i \notin I' \end{cases}, \tag{13}$$

where $r(i;\mathrm{x}) := \sum_{j \in J} \left[\sum_{k \in J} |x(i-j+k)|^2 \right]^{-1/2}$ and with the initialization $\mathrm{x}^{(0)} = \mathrm{y}$.

For the de-noising problem, y is the noisy data, so it is unlike that $y(i) = 0$ for any i. We just consider the case where $x^{(0)}(i) \neq 0$, so $r(i; \mathrm{x}^{(k)}) > 0$, $y(i) / \left[1 + \lambda r(i; \mathrm{x}^{(k)})\right]$ lies strictly between zero and $y(i)$. When the group size K=1 and $R(\mathrm{x}) = |\,\mathrm{x}\,|$, then the solution is obtained by soft thresholding. The computational complexity of each iteration in OGS algorithm is of order *KN*, and the memory required for the algorithm is *2N + K.*

It can be seen from the above OGS thresholding iteration formula (13), the most important parameters are the regularization parameter λ, group size K and the number of iterations.

3.1. The Parameters Set in OGS

The regularization parameter λ is the most important parameter for the de-noising effect in OSG thresholding. λ should be chosen large enough to reduce the noise to a sufficiently negligible level, yet no larger so as to avoid unnecessary signal distortion.

In order to set λ so as to reduce the white Gaussian noise to a desired level, the effect of the OGS thresholding on standard white Gaussian noise is investigated. Although there is no explicit formula in OGS thresholding such as the soft thresholding about the output standard deviation σ of the standard Gaussian noise against the thresholding T, the σ can be found by simulation as a function of λ for a fixed group size. In general, the de-noising process is more sensitive to λ for larger group sizes, hence, the choice of λ is more critical.

Table 1 gives a portion values of λ so that OGS thresholding produces an output signal with specified standard deviation σ when the input signal is standard normal Gaussian noise.

Table 1. Parameter λ for standard normal i.i.d. signal

Groups (K)	Output std σ_x			
	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1×1	3.36	4.38	5.24	6.00
1×3	1.16(1.18)	1.46(1.52)	1.60(1.77)	1.64(1.99)
1×5	0.73(0.75)	0.92(0.95)	1.01(1.12)	1.04(1.25)
2×3	0.59(0.61)	0.74(0.77)	0.80(0.89)	0.82(1.01)
3×5	0.29(0.31)	0.32(0.36)	0.35(0.40)	0.36(0.45)
5×5	0.21(0.23)	0.22(0.26)	0.23(0.29)	0.24(0.32)

The first value of each column is obtained by full convergence and the second value is obtained by 25 iterations. In practice, in order to reduce the amount of calculation, we usually choose 25 iterations.

For example, suppose one is using the OGS thresholding with K=5 for de-noising a signal contaminated by white Gaussian noise with standard deviation σ_x, in order to reduce the noise down to 1 % of its original value, one should set $\lambda = 0.75\sigma_x$ if 25 iterations are used in OGS algorithm. Refer to [11] for more details about the OGS algorithm.

4. The Proposed Method

As mentioned above, we propose a novel TI wavelet de-noising method with OGS algorithm. Signals contaminated with white Gaussian noise through the wavelet transform can be well represented by few wavelet coefficients. According to the group characteristics of wavelet coefficients, we choose OGS thresholding function to process the detail coefficients only, in order to preserve the low frequency shapes of the ECG signals (P-wave and T-wave), after which the noise components are significantly reduced. Finally, reconstruct the wavelet coefficients and inverse-shift the de-noised signal. Fig. 1 shows the block diagram of the TI wavelet de-noising scheme.

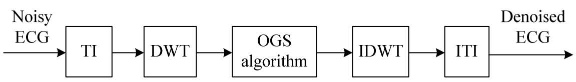


Fig. 1. Block diagram of the proposed de-noising method.

So a possible strategy for de-noising the white Gaussian noise added signals can be built as follows:

1) Determine the suitable wavelet basis function and decomposition layers. Shift the noisy signal within range of cycle spinning to get a new shifted signal and decompose the new signal into wavelet coefficients with DWT.

2) Estimate the noise standard deviation in each layer of the wavelet detail coefficients.

3) Determine the value of K and λ, and the number of iterations in the OGS algorithm. OGS thresholding function is used to shrink the wavelet coefficients of the noisy signal, and then obtain the de-noised signal by inversing discrete wavelet transform.

4) Inverse-shift the de-noised signal to get the original order.

5) Repeat the procedure (1)-(4) many times to get a series of de-noised signals. Calculate the average for all the obtained de-noised signals to get the final de-noised signal.

5. Experiments and Results

In this section, we conduct a number of simulations to evaluate our proposed de-noising method with six representative analog signals and two different ECG signals. The performance of the proposed method is compared with some other conventional methods. In the experiments, the Symlets wavelet (Sym4), 4-level and minimaxi thresholding is adopted. In the process of translation invariant, we take the cycle of 10 times and K=5, $\lambda = 0.75\sigma$, 25 iterations are determined in the OGS algorithm according to Table 1, σ is estimated by different layers of the detail coefficients.

The performance of these methods is evaluated based on the SNR and RMSE. The SNR can be written as follows:

$$SNR_{out} = 10 * \log_{10}\left(\frac{signal\ \ power}{noise\ \ power}\right), \tag{14}$$

RMSE is defined as follows:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(x_i - y_i)^2}{N}}, \tag{15}$$

where x_i is the de-noised signal and y_i is the "clean signal", *N* is the length of the signal.

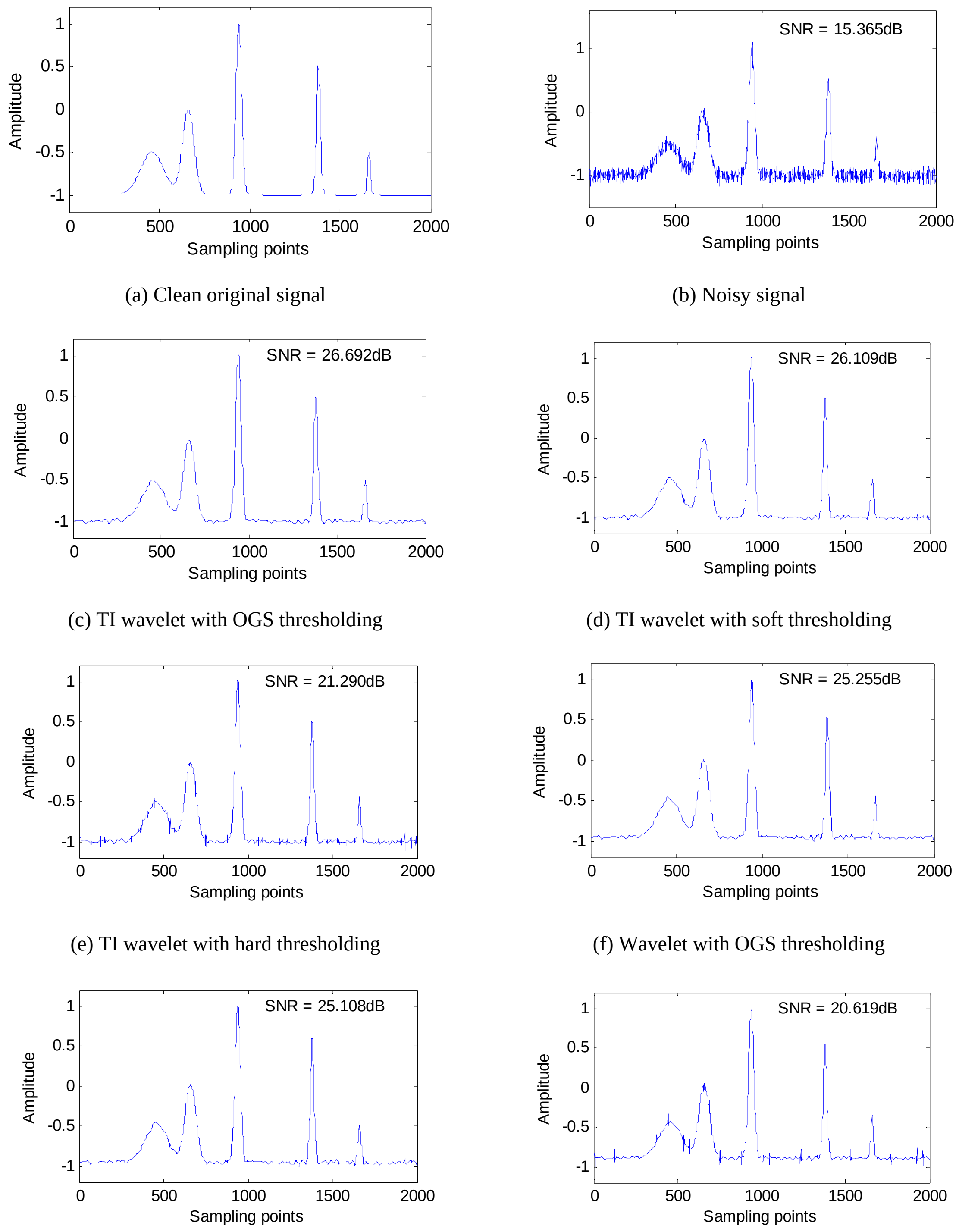
5.1. Analog Signal De-noising

We choose Spikes with 2 000 sampling points as the analog signal to evaluate the performance of the proposed de-noising method. White Gaussian noise with zero mean and standard deviation $\sigma = 0.055$ is added artificially to the Spikes signal resulting in the input SNR=15.365 dB. The "clean signal" and the noisy signal are achieved and shown in Fig. 2(a) and Fig. 2(b) respectively. The traditional DWT and TI wavelet de-noising methods are performed with soft thresholding, hard thresholding and OGS thresholding (Totally six de-noising methods). Fig. 2(c)-Fig. 2(h) shows the de-noised signal using the above six de-noising methods respectively. The output SNR of the proposed method is 26.692 dB, which increases 11.327 dB compared with the input SNR.

It can be observed from Fig. 2 that the above six de-noising methods all can remove the added white Gaussian noise roughly. However, the methods in Fig. 2(e) and Fig. 2(h) deal with the wavelet coefficients by hard thresholding, which may lead to the oscillation in the reconstructed signal, and the soft thresholding is adopted in Fig. 2(d) and Fig. 2(g), which may produce a more smooth reconstructed waveform, but it will reduce the amplitudes whose absolute values are larger than the preset threshold, so a part of the high frequency components of the useful signal will be loss. To overcome the above mentioned disadvantages, we use the new TI wavelet with OGS thresholding de-noising method as proposed. The de-noised signal is shown in Fig. 2(c) which suppresses the Pseudo-Gibbs phenomenon in the signal singularity effectively and gets a higher output SNR.

In order to compare the performance of the six de-noising methods systematically through the input-output SNR, six typical analog signals (Spikes signal, Bumps signal, Doppler signal, Time Shifted Sine signal, Angles signal and Parabolas signal) are adopted in the next experiments.

When the input SNR of the six analog signals ranging from 5 dB to 30 dB for 5 dB per interval increases, the output SNR of different de-noising methods are shown in Fig. 3. The X-axis represents the input SNR and the Y-axis represents the output SNR of the de-noised signal with different de-noising methods.



(a) Clean original signal

(b) Noisy signal

(c) TI wavelet with OGS thresholding

(d) TI wavelet with soft thresholding

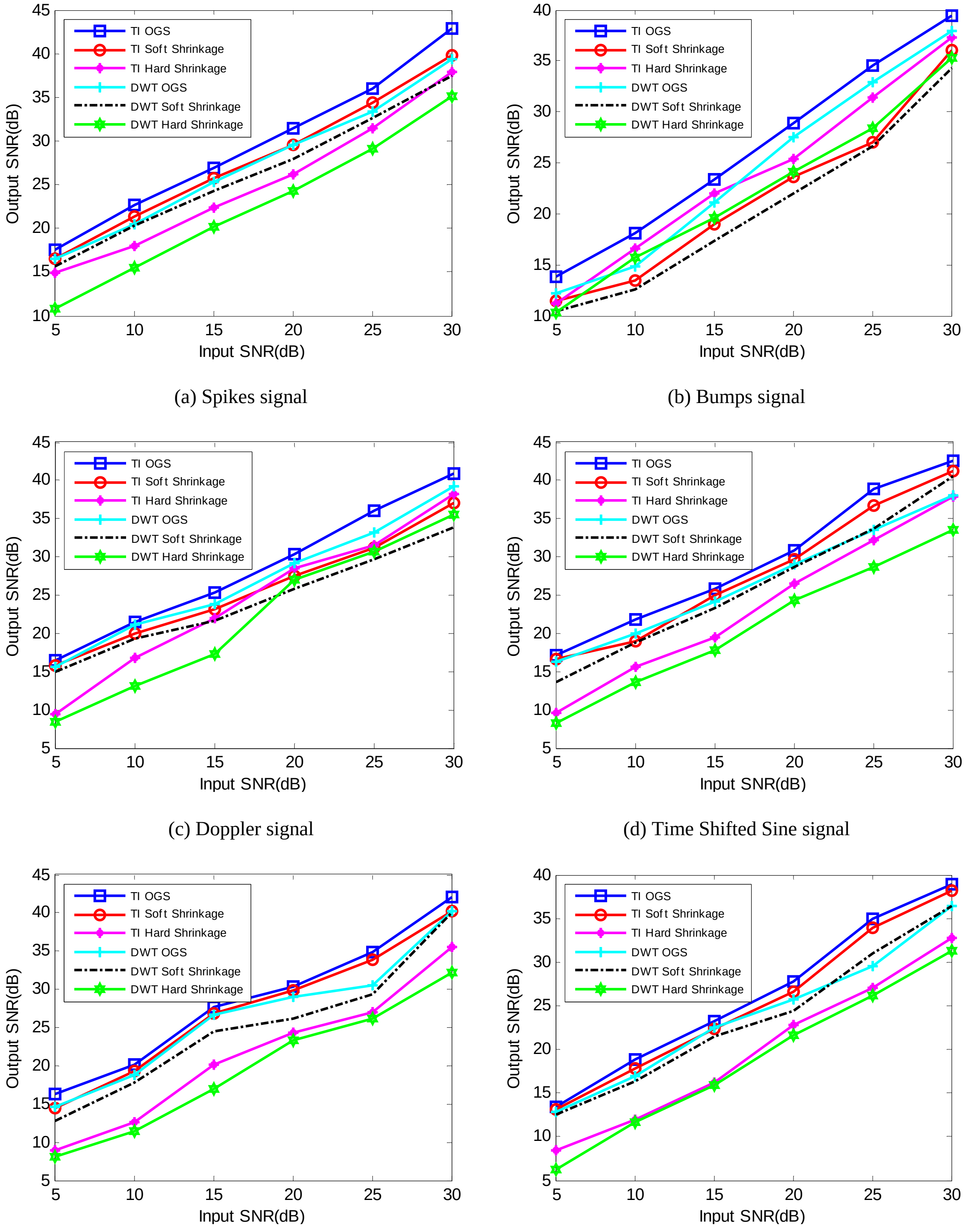
(e) TI wavelet with hard thresholding

(f) Wavelet with OGS thresholding

(g) Wavelet with soft thresholding

(h) Wavelet with hard thresholding

Fig. 2. Spikes signal before and after de-noising with different methods.



(a) Spikes signal (b) Bumps signal

(c) Doppler signal (d) Time Shifted Sine signal

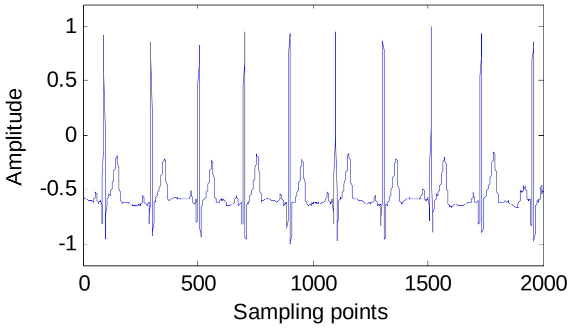
(e) Angles signal (f) Parabolas signal

Fig. 3. Input and output SNR of different analog signals and de-noising methods.

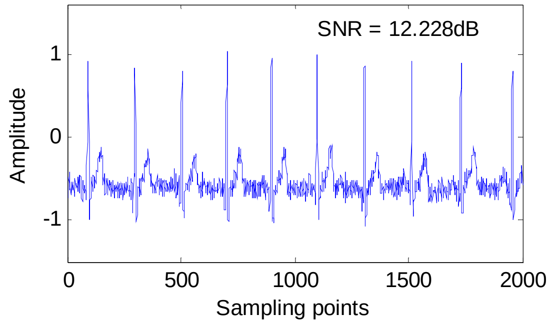
From Fig. 3 we can certainly conclude that for the different analog signals, when the input SNR increases from low to high, the proposed de-noising method is the best of the six methods in terms of the output SNR and shows a stable de-noising performance. As the ECG signal and Spikes signal are similar, the proposed method can also be applied to the ECG signal de-noising.

5.2. ECG Signal De-noising

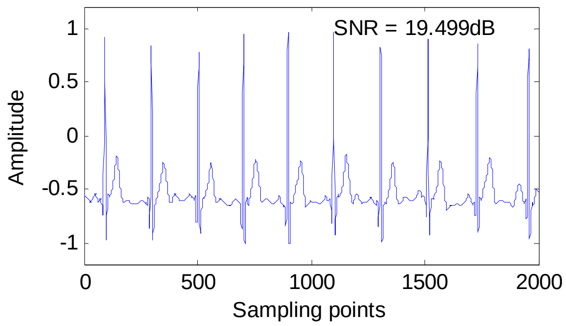
To validate the superiority of the proposed de-noising method, ECG signal in the MIT-BIH database is employed. The length of the original ECG signal is 2 000 sampling points and the sampling rate is 250 Hz. We add the white Gaussian noise with zero mean and standard deviation $\sigma = 0.06$ to the clean original ECG signal, then the input SNR=12.228 dB. The above six methods are adopted for the de-noising experiments respectively. The clean ECG signal, noisy ECG signal and de-noised ECG signal are shown in Fig. 4. The results of the RMSE are given in Table 2. (c), (d), (e), (f), (g) and (h) in Table 2 represents the corresponding de-noising method respectively.



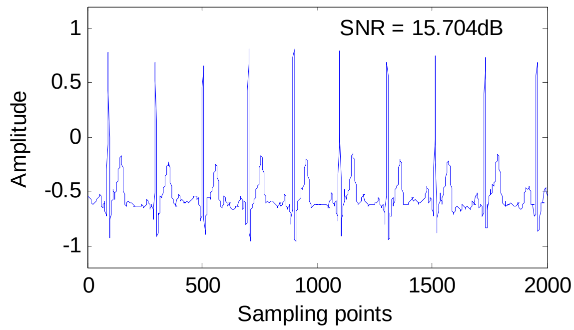
(a) Clean original signal



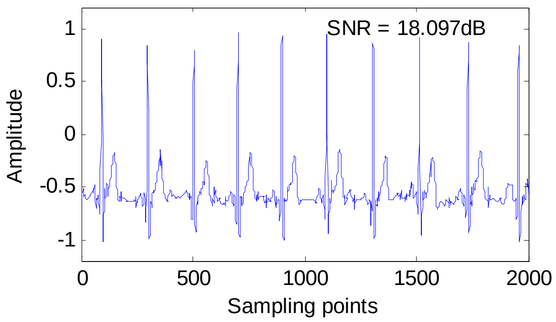
(b) Noisy signal



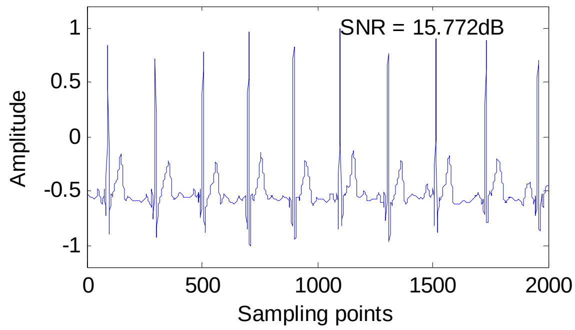
(c) TI wavelet with OGS thresholding



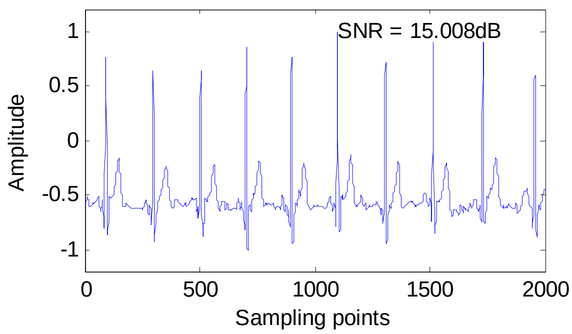
(d) TI wavelet with soft thresholding



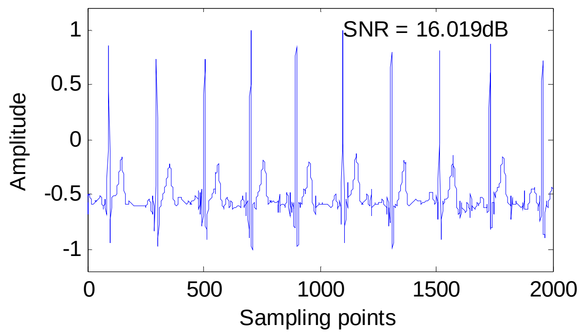
(e) TI wavelet with hard thresholding



(f) Wavelet with OGS thresholding



(g) Wavelet with soft thresholding



(h) Wavelet with hard thresholding

Fig. 4. ECG signal before and after de-noising with different methods.

Table 2. Comparison of different de-noising methods

Methods	(c)	(d)	(e)	(f)	(g)	(h)
RMSE	0.028	0.042	0.036	0.041	0.045	0.046

Seen from Fig. 4, the proposed method not only eliminates the white Gaussian noise effectively, but also suppresses the Pseudo-Gibbs phenomenon in the reconstructed ECG signal, which makes the de-noised ECG signal remain the main characteristics of the original signal and keep the amplitudes of R wave effectively. From Table 2, it is well known that the proposed method performs better than other methods according to the RMSE, the de-noised signal of which is closer to the original clean signal.

The proposed method is also tested on another ECG signal with the same sampling points and sampling frequency. The noisy signal is obtained by adding white Gaussian noise and the input SNR is 12.050 dB. It can be seen from Fig. 5 that the proposed de-noising method has a good performance in preserving the QRS wave and P wave of noisy ECG signal and the oscillation phenomenon is not obvious, which is very important for the detection and diagnosis of cardiovascular disease. The comparison of output SNR and RMSE with other methods are shown in Table 3.

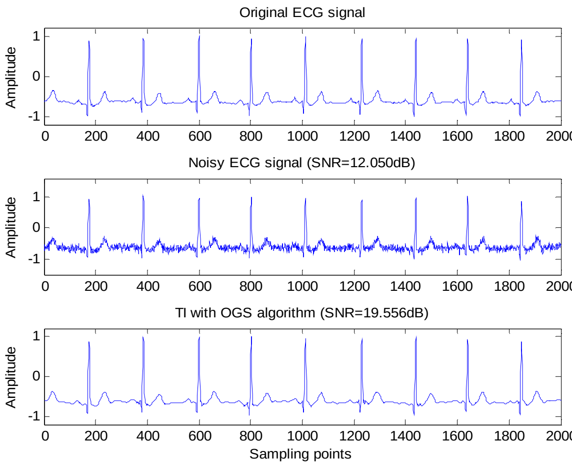


Fig. 5. ECG signal before and after de-noising with the proposed method.

Table 3. Comparison of different de-noising methods.

Methods	(c)	(d)	(e)	(f)	(g)	(h)
SNR	19.56	15.35	18.46	16.84	15.17	16.38
RMSE	0.034	0.041	0.046	0.051	0.058	0.072

From all the above experimental results, we can certainly conclude that the TI wavelet de-noising method with OGS algorithm can be an effective tool for de-noising the ECG signals providing high SNR of the de-noised ECG signals and good visual quality.

6. Conclusions

In the present work, we propose a new ECG signal de-noising method based on the TI wavelet transform and OGS algorithm. The traditional wavelet de-noising method may produce Pesudo-Gibbs phenomenon which related to the signal singularity locations. TI wavelet transform can suppress the phenomenon by cycling spinning the positions of the signal singularity. OGS thresholding function can shrink the wavelet coefficients well.

The experimental results of the analog signals and ECG signals show that the proposed method in this paper is superior to the other traditional de-noising methods in many aspects such as the smoothness, remaining the main ECG geometrical characteristics, which contain valuable physiological information for diagnostic purpose. The proposed de-noising method may be useful for the doctors to accurately diagnose cardiovascular ailments in patients.

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