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Hemispherical Resonator Gyroscope Accuracy Analysis Under Temperature Influence

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Abstract: Frequency splitting of hemispherical resonator gyroscope will change as system operating temperature changes. This phenomenon leads to navigation accuracy of hemispherical resonator gyroscope reduces. By researching on hemispherical resonator gyroscope dynamical model and its frequency characteristic, the frequency splitting formula and the precession angle formula of gyroscope vibrating mode based on hemispherical resonator gyroscope dynamic equation parameters are derived. By comparison, gyroscope precession angle deviation caused by frequency splitting can be obtained. Based on analysis of temperature variation against gyroscope resonator, the design of hemispherical resonator gyroscope feedback controller under temperature variation conditions is researched and the maximum theoretical fluctuation of gyroscope dynamical is determined by using a numerical analysis example.

Keywords: Hemispherical resonator gyroscope, Resonator, Frequency cracking, Temperature, Accuracy analysis.

1. Introduction

As an inertial navigation sensor, hemispherical resonator gyroscope (HRG) has features of long service life, high reliability and low power dissipation, and it has been widely used in fields such as petroleum drilling, space and geophysical exploration [1, 2]. In theory research, Pavlovskii studied on nonideal HRG dynamics [3]. Michal Mrozowsk [4] and Gao [5] researched HRG features by using finite element analysis. Zhbanov Y. K. researched HRG vibration characteristics [6, 7]. Li analyzed the HRG error caused by resonator's Q-factor nonuniformity [8] and Zhao studied on the influence of HRG density's nonuniformity [9]. But all these researches are built on the condition that HRG is working in a stable environment. In practical application, in fact, we find that HRG output is influenced by temperature factor to a large extent. So in this paper, we research on dynamical equation of HRG, derive formula of dynamical equation parameters varying to gyroscope frequency splitting and analyze the reason of dynamical equation parameters varying to temperature change. And then we research on determination method of HRG system feedback controller and dynamical equation parameters fluctuation range under steady operation condition.

2. HRG Dynamical Equation and Frequency Splitting

HRG and its resonator are shown in Fig. 1. When gyroscope resonator is stimulated and the gyroscope is making entirety rotation at speed of Ω, vibrating mode of ideal resonator may make reverse movement against gyroscope body at speed of $k\Omega$. K is a function related to structure constant of ideal resonator and vibrating mode and it is not affected by external conditions change. By the measurement of procession angle of vibrating mode, we can work out the procession angle of the whole gyroscope system. Using differential calculus, the rate of angular motion is obtained. Thus gyroscope basic functions are realized.

Fig. 1. HRG structure diagram

Under the condition of location stimulation, equation of ideal resonator second order inherent vibration type is:

$$\begin{cases} \ddot{p}(t) - c\Omega\dot{q}(t) + \omega_0^2 p(t) = U_1 \\ \ddot{q}(t) + c\Omega\dot{p}(t) + \omega_0^2 q(t) = U_2 \end{cases}, \quad (1)$$

where $p(t)$, $q(t)$ are the nodal displacement of system, $\dot{p}(t)$, $\dot{q}(t)$ are the nodal speed of system, $\ddot{p}(t)$, $\ddot{q}(t)$ are the accelerated speed of system, U_1 and U_2 are the stimulations of system.

On this condition, the precession angle of HRG system is completely proportion to precession angle of resonator vibrating mode. There is only one resonant frequency in the ideal resonator. According to Fan Shangchun's theory this ideal resonant frequency is:

$$\omega_0 = \frac{6}{r^2}\sqrt{\frac{E \cdot u(\varphi)}{3(1+\mu)\rho \cdot v(\varphi)}}, \quad (2)$$

where

$$u(\varphi) = \int_{\varphi_0}^{\varphi_F} \frac{\tan^4 \frac{\varphi}{2}}{\sin^3 \varphi} h^3(\varphi) d\varphi$$

$$v(\varphi) = \int_{\varphi_0}^{\varphi_F} (5 + \sin^2 \varphi + 4\cos\varphi)\sin\varphi \cdot \tan^4 \frac{\varphi}{2} h(\varphi) d\varphi,$$

ϕ_0, ϕ_F are the terminal angles and tip angle of hemispherical shell respectively, $r, h(\varphi)$ are the radius and wall thickness of hemispherical shell respectively, E, ρ, μ are the Young's modulus, density and Poisson's ratio of hemispherical shell respectively.

In actual production of HRG, due to inadequate working accuracy of all production processes and error accumulation, we can't obtain ideal hemispherical resonator with completely uniform quality. This will lead to damping effect and anisoelasticity effect as well as making two resonant frequencies existing in the operation of hemispherical resonator. We call difference value between two resonant frequencies as frequency splitting, which is main error source influencing gyroscope accuracy. In order to obtain formula of frequency splitting and in consideration of damping effect and anisoelasticity effect, we set HRG dynamic equation as follows:

$$\begin{cases} \ddot{p}(t) - D_{xx}\dot{p}(t) - \\ c\Omega\dot{q}(t) + \omega_1^2 p(t) - Z_{xy} q(t) = U_1 \\ \ddot{q}(t) + c\Omega\dot{p}(t) - \\ D_{yy}\dot{q}(t) - Z_{yx} p(t) + \omega_2^2 q(t) = U_2 \end{cases}, \quad (3)$$

where

$$D_{xx} = D_{yy} = -\omega_0^2 \xi$$

D_{xx} and D_{yy} are the equation parameters produced by damping effect, ω_1^2 , Z_{xy} , Z_{yx} and ω_2^2 are the equation parameters affected by anisoelasticity effect of system. Among them, ω_1 and ω_2 are the systemic vibrating frequency along systemic nodal displacement direction. Z_{xy} and Z_{yx} are the vibrating coupling resulted from systemic vibration due to anisoelasticity effect.

For studying frequency splitting character of HRG, it is necessary to change dynamic equation into state space equation. State variables are selected as follows:

$$x_1(t) = \dot{p}(t), x_2(t) = \dot{q}(t),$$
$$x_3(t) = p(t), x_4(t) = q(t),$$

It is convenient to define $\boldsymbol{x}(t)$

$$\boldsymbol{x}(t) \triangleq [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^{\mathrm{T}},$$

And the original dynamical equation is changed as the following state space form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad (4)$$

where

$$\boldsymbol{A}=\begin{bmatrix} D_{xx} & c\Omega & -\omega_1^2 & Z_{xy} \\ -c\Omega & D_{yy} & Z_{xy} & -\omega_2^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{B}=\begin{bmatrix} U_1 \\ U_2 \\ 0 \\ 0 \end{bmatrix}$$

Equation (4) is systemic state equation. And characteristic equation of the system is:

$$|\lambda\boldsymbol{I}-\boldsymbol{A}|= \begin{vmatrix} \lambda-D_{xx} & -c\Omega & \omega_1^2 & -Z_{xy} \\ c\Omega & \lambda-D_{yy} & -Z_{yx} & \omega_2^2 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix}=0 \quad (5)$$

$$\lambda^2(\lambda-D_{xx})(\lambda-D_{yy})+\omega_1^2\omega_2^2+ \omega_2^2\lambda(\lambda-D_{xx})+\omega_1^2\lambda(\lambda-D_{yy})=0$$

$$\lambda^4-(D_{xx}+D_{yy})\lambda^3+\omega_1^2\omega_2^2 (D_{xx}D_{yy}+\omega_1^2+\omega_2^2)\lambda^2- (\omega_2^2D_{xx}+\omega_1^2D_{yy})\lambda+=0 \quad (6)$$

We define

$$\lambda=\lambda_1+\frac{1}{4}(D_{xx}+D_{yy})$$

$$s_2=-\frac{3}{8}(D_{xx}+D_{yy})^2+ (D_{xx}D_{yy}+\omega_1^2+\omega_2^2)$$

$$s_1=-\frac{1}{16}(D_{xx}+D_{yy})^3+ \frac{1}{2}(D_{xx}+D_{yy})(D_{xx}D_{yy}+\omega_1^2+\omega_2^2)- (\omega_2^2D_{xx}+\omega_1^2D_{yy})$$

$$s_0=-\frac{1}{128}(D_{xx}+D_{yy})^4+ \frac{1}{16}(D_{xx}+D_{yy})^2(D_{xx}D_{yy}+\omega_1^2+\omega_2^2)- \frac{1}{4}(\omega_2^2D_{xx}+\omega_1^2D_{yy})(D_{xx}+D_{yy})+ \omega_1^2\omega_2^2$$

From (6) we have

$$\lambda_1^4+s_2\lambda_1^2+s_1\lambda_1+s_0=0 \quad (7)$$

Solving the equation (7), we have

$$\lambda=\frac{1}{4}(D_{xx}+D_{yy})\pm j\sqrt{\frac{s_2+s_1\pm\sqrt{(s_1+s_2)^2-4s_0}}{2}} \quad (8)$$

From (8), there are two resonant frequency of HRG in vibration process

$$\omega_l=\sqrt{\frac{s_2+s_1-\sqrt{(s_1+s_2)^2-4s_0}}{2}}$$

$$\omega_h=\sqrt{\frac{s_2+s_1+\sqrt{(s_1+s_2)^2-4s_0}}{2}}$$

The difference value of resonant frequency is considered to be frequency splitting of hemispherical resonator.

$$\Delta=\sqrt{\frac{s_2+s_1+\sqrt{(s_1+s_2)^2-4s_0}}{2}}- \sqrt{\frac{s_2+s_1-\sqrt{(s_1+s_2)^2-4s_0}}{2}} \quad (9)$$

3. Influence of HRG Frequency Splitting

In this section, we will research on the influence of hemispherical resonator frequency splitting.

Suppose that HRG works in force feedback mode, and resonator marginal radial vibration can be described:

$$\begin{aligned} w(\varphi,t) &= p(t)\cos 2\varphi+q(t)\sin 2\varphi \\ &=[a(t)\cos\omega t+m(t)\sin\omega t]\cos 2\varphi+ \\ &\quad [b(t)\cos\omega t+n(t)\sin\omega t]\sin 2\varphi \end{aligned}$$

When gyroscope resonator is ideal, vibrating mode angle of HRG is:

$$\tan\psi=\frac{\sqrt{n^2+b^2}}{\sqrt{m^2+a^2}} \quad (10)$$

When there is frequency splitting in hemispherical resonator vibration, resonator marginal radial vibrations produced by ω_l and ω_h are as follows:

$$\begin{aligned} w_1(\varphi,t) = {} & [a(t)\cos\omega_l t + \\ & m(t)\sin\omega_l t]\cos 2\varphi + \\ & [b(t)\cos\omega_l t + \\ & n(t)\sin\omega_l t]\sin 2\varphi \end{aligned} \quad (1)$$

$$\begin{aligned} w_2(\varphi,t) = {} & [a(t)\cos\omega_h t + \\ & m(t)\sin\omega_h t]\cdot\cos 2(\varphi-\varphi_1) + \\ & [b(t)\cos\omega_h t + \\ & n(t)\sin\omega_h t]\sin 2(\varphi-\varphi_1) \end{aligned}$$

where $2\varphi_1$ is the angle between two vibration shafts. In order to make it easy to calculate, we set $\varphi_1 = \frac{\pi}{4}$, then:

$$\begin{aligned} w_2(\varphi,t) = {} & [a\cos\omega_h t + m\sin\omega_h t]\sin 2\varphi - \\ & [b\cos\omega_h t + n\sin\omega_h t]\cos 2\varphi \end{aligned}$$

The total compounded vibration locus is

$$\begin{aligned} w(\varphi,t) = {} & [a\cos\omega_l t + m\sin\omega_l t - \\ & b\cos\omega_h t - n\sin\omega_h t]\cos 2\varphi + \\ & [b\cos\omega_l t + n\sin\omega_l t + \\ & a\cos\omega_h t + m\sin\omega_h t]\sin 2\varphi \end{aligned}$$

Suppose that $\omega_l < \omega_c < \omega_h$, $\Delta_1 = \omega_c - \omega_l$, $\Delta_2 = \omega_h - \omega_c$, $\Delta = \omega_h - \omega_l$,

$$\begin{aligned} w(\varphi,t) = {} & \{[a\cos\Delta_1 t - m\sin\Delta_1 t - \\ & b\cos\Delta_2 t - n\sin\Delta_2 t]\cos\omega_c t + \\ & [a\sin\Delta_1 t + m\cos\Delta_1 t + \\ & b\sin\Delta_2 t - n\sin\Delta_2 t]\cdot \\ & \sin\omega_c t\}\cos 2\varphi + \\ & \{[b\cos\Delta_1 t - n\sin\Delta_1 t + \\ & a\cos\Delta_2 t + m\sin\Delta_2 t]\cos\omega_c t + \\ & [b\sin\Delta_1 t + n\cos\Delta_1 t - \\ & a\sin\Delta_2 t + m\cos\Delta_2 t]\cdot \\ & \sin\omega_c t\}\sin 2\varphi \end{aligned}$$

Let

$$\begin{aligned} a_2 = {} & a\cos\Delta_1 t - m\sin\Delta_1 t - \\ & b\cos\Delta_2 t - n\sin\Delta_2 t \end{aligned}$$

$$\begin{aligned} m_2 = {} & a\sin\Delta_1 t + m\cos\Delta_1 t + \\ & b\sin\Delta_2 t - n\sin\Delta_2 t \end{aligned}$$

$$\begin{aligned} b_2 = {} & b\cos\Delta_1 t - n\sin\Delta_1 t + \\ & a\cos\Delta_2 t + m\sin\Delta_2 t \end{aligned}$$

$$\begin{aligned} n_2 = {} & b\sin\Delta_1 t + n\cos\Delta_1 t - \\ & a\sin\Delta_2 t + m\cos\Delta_2 t \end{aligned}$$

Define determinant $|\boldsymbol{P}|$ as follows:

$$\begin{aligned} |\boldsymbol{P}| & = \begin{vmatrix} n_2 & b_2 \\ m_2 & a_2 \end{vmatrix} \\ & = (a^2 + m^2 + b^2 + n^2)\sin\Delta t \end{aligned}$$

Suppose

$$I(\boldsymbol{P}) = n_2^2 + b_2^2 + m_2^2 + a_2^2$$

$$\alpha = \frac{\det\boldsymbol{P}}{I(\boldsymbol{P})} = \frac{\det\boldsymbol{P}}{n_2^2 + b_2^2 + m_2^2 + a_2^2}$$

Then

$$I(\boldsymbol{P}) = 2(a^2 + m^2 + b^2 + n^2)$$
$$\alpha = \frac{\det\boldsymbol{P}}{I(\boldsymbol{P})} \approx \frac{1}{2}\Delta t,$$

The coefficient matrix of hemispherical resonator vibration main standing wave can be expressed as:

$$\begin{aligned} \boldsymbol{P}_m & = \boldsymbol{P} + \alpha\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\boldsymbol{P}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ & = \begin{bmatrix} n_2 - \alpha a_2 & b_2 + \alpha m_2 \\ m_2 + \alpha b_2 & a_2 - \alpha n_2 \end{bmatrix} \end{aligned}$$

And gyroscope vibrating mode precession angle can be expressed as:

$$\begin{aligned} \tan\psi_1 & = \frac{\sqrt{(n_2 - \alpha a_2)^2 + (b_2 + \alpha m_2)^2}}{\sqrt{(m_2 + \alpha b_2)^2 + (a_2 - \alpha n_2)^2}} \\ & \approx [(m + n - \frac{3}{2}a\Delta t + \frac{1}{2}b\Delta t)^2 + \\ & \quad (a + b + \frac{3}{2}m\Delta t - \frac{1}{2}n\Delta t)^2]^{\frac{1}{2}}\cdot \\ & \quad [(m - n + \frac{1}{2}a\Delta t + \frac{3}{2}b\Delta t)^2 + \\ & \quad (a - b - \frac{1}{2}m\Delta t - \frac{3}{2}n\Delta t)^2]^{-\frac{1}{2}} \end{aligned} \quad (11)$$

Compared with (10) and (11), we notice that vibrating mode precession angle is influenced by frequency splitting and affect gyroscope output accuracy.

When the working environment of gyroscope is maintaining stability, all elements of gyroscope state space matrix don't change. Therefore gyroscope frequency splitting formula composed of these factors also keeps stable and makes gyroscope bias output stable, which is shown in Fig. 2.

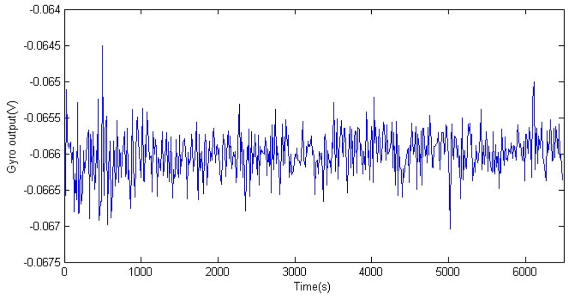

Fig. 2. HRG steady bias output.

When external temperature changes, as for hemispherical resonator, we re-analyze formula (2).

We suppose d as central studdle radius of hemispherical resonator, for radius r and wall thickness $h(\varphi)$, we have

$$\delta = r + h + d\,,$$

where δ is the constant value in design of hemispherical resonator.

Suppose that dilatation coefficient of quartz material is γ, then

$$\begin{aligned} h(T) &= h_0[1+\gamma(T-T_0)] \\ d(T) &= d_0[1+\gamma(T-T_0)] \end{aligned},$$

where h_0, d_0 are the initial thickness and initial radius. T_0 is initial temperature.

$h(T)$ is increased with rising of temperature, $r(T)$ is decreased with rising of temperature, and the resonator density ρ reduces with thermal expansion of hemispherical resonator.

For Young's modulus E and temperature, we have the following formula:

$$\frac{E-E_0}{T} = -b\exp\left(-\frac{T_1}{T}\right)$$

where E_0, b, T_1 are the empirical constants.

This formula indicates that Young's modulus E will reduce with rising of temperature.

From the above analysis, we can conclude that the parameters involved with calculation of hemispherical resonant frequency will change with temperature, so the resonant frequency will change with temperature too. With variations in temperature, actual hemispherical resonator produces new splitting decomposition based on original basis to make output of HRG changes.

We can verify the above-mentioned analysis of frequency splitting through some temperature tests. In Fig. 3, it is gyroscope bias output with temperature changing range of 25 °C-55 °C. After responding to temperature variation, gyroscope forms stable bias output, which indicates that hemispherical resonator has stable new frequency splitting value.

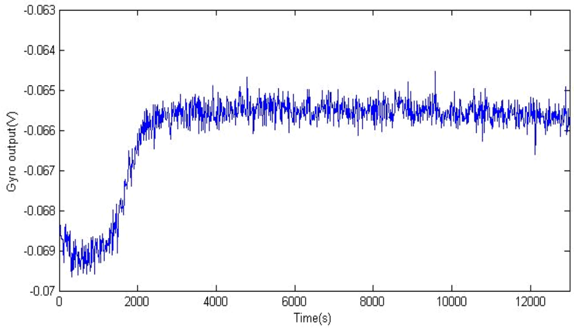

Fig. 3. HRG steady bias output with temperature changing.

However, in some temperature test of wider range (For example, 25 °C-65 °C), HRG fails to form stable output. The gyroscope system has not been steady, see Fig. 4.

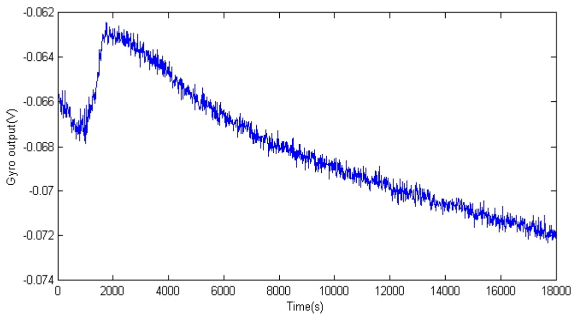

Fig. 4. HRG unsteady bias output with temperature changing.

This indicates that as to external temperature change, parametric variation of hemispherical gyroscope system dynamical equation parameter variations existing steady limit. How to obtain systemic parameters variation section through known dynamical equation and other parameters? This question is what we discuss and study in the later section.

4. Fluctuation of Gyroscope Dynamical Equation Parameters

For HRG resonator operating in steady operating section, we suppose dynamical equation with parametric variation:

$$\begin{cases} \ddot{p}(t) - (D_{xx} + e_{11})\dot{p}(t) - c\Omega\dot{q}(t) + \\ (\omega_1^2 + e_{13})p(t) - (Z_{xy} + e_{14})q(t) = U_1 \\ \ddot{q}(t) + c\Omega\dot{p}(t) - (D_{yy} + e_{22})\dot{q}(t) - \\ (Z_{yx} + e_{23})p(t) + (\omega_2^2 + e_{24})q(t) = U_2 \end{cases}$$

State space expression corresponding to system is

$$\boldsymbol{A} + \boldsymbol{E} = \begin{bmatrix} D_{xx} + e_{11} & c\Omega & -\omega_1^2 + e_{13} & Z_{xy} + e_{14} \\ -c\Omega & D_{yy} + e_{22} & Z_{xy} + e_{23} & -\omega_2^2 + e_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where

$$\boldsymbol{E} = \begin{bmatrix} e_{11} & 0 & e_{13} & e_{14} \\ 0 & e_{22} & e_{23} & e_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{E}$ is variation matrix of systemic parameter.

In order to guarantee hemispherical gyroscope can form steady output under the condition of parametric change, it is necessary to add feedback controller to hemispherical gyroscope system. This control system diagram as shown in the Fig. 5.

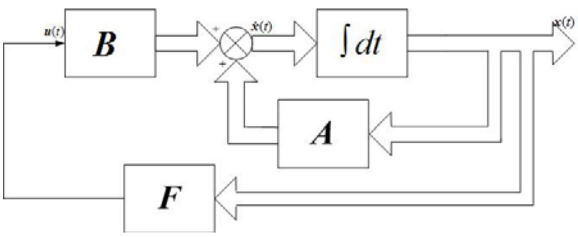

Fig. 5. Control system diagram.

In Fig. 5, systematic input variable $\boldsymbol{u}$ is systematic stimulation of HRG. Output $\boldsymbol{x}$ is speed and displacement of system nodal. The system adjusts state matrix $\boldsymbol{A}$ by changing feedback $\boldsymbol{F}$.

Now the question is how we shall design a feedback controller to meet the systematic expectation closed-loop pole and where is the controller steady limit.

For the control system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

We introduce the following theorem:

Theorem 1. If there is feedback law $\boldsymbol{u} = \boldsymbol{F}\boldsymbol{x}$ making matrix $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ stable, then suppose

$$\varepsilon \overset{\Delta}{=} (\|\boldsymbol{X}^{-1}\| \bullet \|\boldsymbol{X}\|)^{-1} \min_i |\mathrm{Re}(s_i)|, \quad i = 1,2,3,4 \tag{12}$$

where $\boldsymbol{X}$ is the eigenvector of matrix $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$, s_i is the characteristic root of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$.

When parametric variation $\boldsymbol{E}$ meets the requirements

$$\|\boldsymbol{E}\| < \varepsilon \tag{13}$$

The matrix $\boldsymbol{A} + \boldsymbol{E} + \boldsymbol{B}\boldsymbol{F}$ is stable too.

Certification: Because $\boldsymbol{X}$ is eigenvector of matrix $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$, s_i is characteristic root of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$, so we have

$$\|\boldsymbol{X}^{-1}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{X}\| = \mathrm{diag}\{s_1, s_2, s_3, s_4\}$$

According to G. W. Steuart theory, Characteristic values of matrix $\boldsymbol{A} + \boldsymbol{E} + \boldsymbol{B}\boldsymbol{F}$ are in the following intersection of aggregation

$$\mathrm{A}_i = \{s : |s - s_i| \le \|\boldsymbol{X}^{-1}\boldsymbol{E}\boldsymbol{X}\|\}, i = 1,2,3,4$$

When formula (13) holds, for $\|\boldsymbol{X}^{-1}\boldsymbol{E}\boldsymbol{X}\|$ has the following relation:

$$\|\boldsymbol{X}^{-1}\boldsymbol{E}\boldsymbol{X}\| \le \|\boldsymbol{X}^{-1}\| \bullet \|\boldsymbol{X}\| \cdot \varepsilon = \min_i |\mathrm{Re}(s_i)|$$

The stability of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ makes $\mathrm{Re}(s_i) < 0$, so for any s which meets $|s - s_i| \le \|\boldsymbol{X}^{-1}\boldsymbol{E}\boldsymbol{X}\|$, we have

$$\mathrm{Re}(s_i) < 0,$$

It means $\boldsymbol{A} + \boldsymbol{E} + \boldsymbol{B}\boldsymbol{F}$ is stable.

For HRG system, suppose system feedback matrix as follows:

$$\boldsymbol{F} = [f_1 \quad f_2 \quad f_3 \quad f_4],$$

So

$$\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$w_{11} = D_{xx} + U_1 f_1$$
$$w_{12} = D_{xy} + c\Omega + U_1 f_2$$
$$w_{13} = -\omega_1^2 + U_1 f_3$$
$$w_{14} = Z_{xy} + U_1 f_4$$
$$w_{21} = D_{yx} - c\Omega + U_2 f_1$$
$$w_{22} = D_{yy} + U_2 f_2$$
$$w_{23} = Z_{yx} + U_2 f_3$$
$$w_{24} = -\omega_2^2 + U_2 f_4$$

And

$$\begin{aligned}&|rI - A - BF| \\ &= r^4 - (D_{xx} + D_{yy} + U_1 f_1 + U_2 f_2) r^3 + \\ &[(D_{xx} + U_1 f_1)(D_{yy} + U_2 f_2) + \\ &(\omega_2^2 - U_2 f_4 + \omega_1^2 - U_1 f_3)] r^2 - \\ &[(D_{xx} + U_1 f_1)(\omega_2^2 - U_2 f_4) + \\ &(D_{yy} + U_2 f_2)(\omega_1^2 - U_1 f_3)] r + \\ &(\omega_2^2 - U_2 f_4)(\omega_1^2 - U_1 f_3)\end{aligned}$$

Suppose the anticipant closed-loop pole of system is: $s_i, i = 1,2,3,4$, then

$$\begin{aligned}&|rI - A - BF| \\ &= r^4 - (s_1 + s_2 + s_3 + s_4) r^3 + \\ &[s_1 s_2 + (s_1 + s_2)(s_3 + s_4) + s_3 s_4] r^2 - \\ &[s_3 s_4 (s_1 + s_2) + s_1 s_2 (s_3 + s_4)] r + \\ &s_1 s_2 s_3 s_4\end{aligned}$$

Then we have

$$\begin{aligned}&s_1 + s_2 + s_3 + s_4 \\ &= D_{xx} + D_{yy} + U_1 f_1 + U_2 f_2 \\ &s_1 s_2 + (s_1 + s_2)(s_3 + s_4) + s_3 s_4 \\ &= (D_{xx} + U_1 f_1)(D_{yy} + U_2 f_2) + \\ &\quad (\omega_2^2 - U_2 f_4 + \omega_1^2 - U_1 f_3) \\ &s_3 s_4 (s_1 + s_2) + s_1 s_2 (s_3 + s_4) \\ &= (D_{xx} + U_1 f_1)(\omega_2^2 - U_2 f_4) + \\ &\quad (D_{yy} + U_2 f_2)(\omega_1^2 - U_1 f_3) \\ &s_1 s_2 s_3 s_4 = (\omega_2^2 - U_2 f_4)(\omega_1^2 - U_1 f_3)\end{aligned}$$

By solving the above equation group, we can obtain the systematic feedback matrix F.

Add feedback matrix solved to systematic feature equation, we obtain eigenvector of this pole:

At last, we can obtain systematic eigenvector matrix $X = (X_1 \quad X_2 \quad X_3 \quad X_4)$ and X^{-1}, we add it to formula（12）to obtain maximum parameter changing section ε allowed by this system, we will make further instruction by using value simulation.

Suppose

$$A = \begin{bmatrix} -3.3\times10^{-7} & -1.2418 & -8999999.142 & 0 \\ -1.2418 & -3.3\times10^{-7} & 0 & -8999999.1418 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} U_1 \\ U_2 \\ 0 \\ 0 \end{bmatrix}$$

The theoretical frequency splitting is 4.1×10^{-8} Hz, suppose the anticipant closed-loop pole of system is:

$$s_1 = -2 + j2\sqrt{3},\ s_2 = -2 - j2\sqrt{3},$$
$$s_3 = -10,\ s_4 = -10$$

We can have

$$F = [24 \quad 72475279.1 \quad -8999999.14 \quad 1739409912]$$

Suppose X is eigenvector of matrix $A + BF$, where

$$X = (X_1, X_2, X_3, X_4)$$

X_1, X_2, X_3, X_4 are corresponding to s_1, s_2, s_3, s_4,

Finally we have

$$\begin{aligned}\varepsilon &\overset{\Delta}{=} (\|X^{-1}\| \bullet \|X\|)^{-1} \min_i |\mathrm{Re}(\lambda_i)|, i = 1,2,3,4 \\ &= \frac{2}{0.5025} = 3.9801\end{aligned} \quad (1)$$

This ε is the parametrical variation range for gyroscope, when parametric variation E meets the condition of $\|E\| < 3.9801$, systematic output is still stable. Meanwhile, ε also describes the sensibility of gyroscope against parametrical change. Add ε to frequency splitting formula to obtain frequency splitting of $\omega_\Delta = 4.1667\times10^{-5}$ in maximum variation limit. Compared to the theoretical frequency splitting we have before, this frequency indicates that system is relatively sensitive to parametrical change.

5. Conclusions

This paper obtains frequency splitting formula by research on HRG dynamical equation and state space equation. Based on this, we obtain why HRG output changes with temperature and the gyroscope has maximum disturbance limit in stable operation section. Its significance is that direct contact systematic parameters with parametrical variation, and thus to offer limit of parametrical range of variation. Although it has conservative, it has relatively high actual significance.

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