

The Structures of Fuzzifying Measure

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Abstract: A new so-called fuzzifying measurable theory that generalizes the classical measurable theory is established and the structures of such new theory are discussed very detailed. In the last, we study the product of two fuzzifying measures and consider a problem, which is like the third of the open problems in fuzzy measure presented by Z. Wang. We have solved this problem satisfactorily in the new theory. Copyright © 2014 IFSA Publishing, S. L.

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1. Introduction

Since M. Sugeno [10] introduced the concept of fuzzy measure in 1974, the study of fuzzy measure theory has gained rich conclusions [5-9]. But most of them are concentrated on single-valued functions. Motivated by Aumann's integral of set-valued functions, Caimei Guo, Deli Zhang etc. introduced fuzzy integral of set-valued functions and then established on set-valued fuzzy measure [4]. In their paper, the fuzzy measure $\tilde{m}$ pertains to the set noted by $P_l(F_l(X))$ ($F_l(X) = \{\tilde{A}; \tilde{A}: X \rightarrow L\}$, L is a complete residual lattice or $L = P_0(R^+)$, $I = (0,1)$).

In 1991, Ying Ming-Sheng used a semantic method of continuous-valued logic LX to propose the concept of fuzzifying topology [1-2]. In 1993, Shen Ji-Zhong uses the same method to establish the theory of fuzzifying groups and gain a lot of good algebraic properties [3]. They all successfully extended the application of such theory. Using the same method, fuzzifying measure $\tilde{m} \in F_l(P(X))$ and fuzzifying measure space have been established [13,

14] and being proved useful in many applying domain such as fuzzy control, finance model etc. But being a new theory, the structure of such theory is not clarified. In this article, such difficult problem has been studied and some satisfaction results have been gained.

First, we display the fuzzy logical and corresponding set-theoretical notations used in this paper.

$$\begin{aligned} 1) \quad & [\neg\varphi] := 1 - [\varphi]; \\ & [\varphi \wedge \psi] := \min([\varphi], [\psi]); \\ & [\varphi \rightarrow \psi] = [\varphi] \alpha [\psi] := \min(1, 1 - [\varphi] + [\psi]); \\ & [\varphi \leftrightarrow \psi] = [\varphi] \beta [\psi]; \quad [(\forall x)\varphi(x)] = \inf_{x \in X} [\varphi(x)]; \\ & [(\exists x)\varphi(x)] = \sup_{x \in X} [\varphi(x)], \end{aligned}$$

where X is the universe of discourse,
 $[x \in \tilde{A}] := \tilde{A}(x)$.

$$\begin{aligned} 2) \quad & \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi); \\ & \varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi); \\ & \tilde{A} \subseteq \tilde{B} := (\forall x)(x \in \tilde{A} \rightarrow x \in \tilde{B}); \end{aligned}$$

$$\tilde{A} \equiv \tilde{B} := (\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A}).$$

$$3) [a] = a, \text{ if } a \in [0, 1].$$

Then, some definitions in classic measure theory [15] and in fuzzifying measurable theory are all display.

Definition 1.1. Let X is universe, $\mathfrak{R} \subseteq P(X)$, satisfy:

$$1) X \in \mathfrak{R};$$

$$2) A \in \mathfrak{R} \Rightarrow A^c \in \mathfrak{R};$$

$$3) \{A_n, n \geq 1\} \subseteq \mathfrak{R} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{R}.$$

Then $\mathfrak{R}$ is called σ -algebra and $(X, \mathfrak{R})$ a measurable space.

Definition 1.2. Let X is a non-empty set and $\mathfrak{R}$ is a σ -algebra of X . A set function $m: \mathfrak{R} \rightarrow [0, 1]$ is called a normal strong semi-measure if

$$1) m(\emptyset) = 0;$$

$$2) A, B \in \mathfrak{R} \text{ and } A \subseteq B \Rightarrow m(A) \leq m(B);$$

$$3) \{A_n, n \geq 1\} \subseteq \mathfrak{R} \text{ and}$$

$$A_n \subseteq A_{n+1} \Rightarrow m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} m(A_n);$$

$$3') \{A_n, n \geq 1\} \subseteq \mathfrak{R} \text{ and}$$

$$A_{n+1} \subseteq A_n \Rightarrow m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} m(A_n).$$

Definition 1.3. Let X is the universe, $\tilde{\mathfrak{R}}$ is a mapping from $P(X)$ to $I=[0,1]$ ($\tilde{\mathfrak{R}} \in F_I(P(X))$) satisfy the following conditions:

$$1) \vdash X \in \tilde{\mathfrak{R}};$$

$$2) \vdash (\forall A)(A \in \tilde{\mathfrak{R}} \rightarrow A^c \in \tilde{\mathfrak{R}});$$

$$3) \text{ For any } \{A_n, n \in N\},$$

$$\vdash (\forall n)(n \in N \rightarrow A_n \in \tilde{\mathfrak{R}}) \rightarrow \bigcup_{n \in N} A_n \in \tilde{\mathfrak{R}}.$$

Then $\tilde{\mathfrak{R}}$ is called a fuzzifying σ -algebra and $(X, \tilde{\mathfrak{R}})$ a fuzzifying measurable space.

If (3) instead of (3'), then $\tilde{\mathfrak{R}}$ is called a fuzzifying algebra [13].

$$(3') \text{ For any } \{A_n, n \in N_k\},$$

$$\vdash (\forall n)(n \in N_k \rightarrow A_n \in \tilde{\mathfrak{R}}) \rightarrow \bigcup_{n \in N_k} A_n \in \tilde{\mathfrak{R}},$$

$$N_k = \{1, 2, \dots, k\}.$$

Definition 1.4. Let $(X, \tilde{\mathfrak{R}})$ is a fuzzifying measurable space. $\tilde{m}$ is a mapping from $P(X)$ to I ($\tilde{m} \in F_I(P(X))$), the sign $\lim(A_n \in \tilde{m})$ is denoted as: $\lim(A_n \in \tilde{m}) :=$

$$(\exists n)(\forall m)((n \in N) \wedge (m \geq n) \rightarrow A_m \in \tilde{m}),$$

and satisfy the following conditions:

$$1) \vdash (\neg(\emptyset \in \tilde{m})) \wedge (X \in \tilde{m});$$

$$2) \vdash (\forall A)(\forall B)((A \in \tilde{\mathfrak{R}}) \wedge (B \in \tilde{\mathfrak{R}}) \wedge (A \subseteq B) \rightarrow (A \in \tilde{m} \rightarrow B \in \tilde{m}));$$

$$3) \text{ For any } \{A_n, n \in N\}$$

$$\vdash (\forall A_n)((A_n \in \tilde{\mathfrak{R}}) \wedge (A_n \subseteq A_{n+1}) \rightarrow ((\bigcup_{n=1}^{\infty} A_n \in \tilde{m}) \leftrightarrow \lim(A_n \in \tilde{m})));$$

$$3') \text{ For any } \{A_n | n \in N\}$$

$$\vdash (\forall A_n)((A_n \in \tilde{\mathfrak{R}}) \wedge (A_{n+1} \subseteq A_n) \rightarrow ((\bigcap_{n=1}^{\infty} A_n \in \tilde{m}) \leftrightarrow \lim(A_n \in \tilde{m}))).$$

Then $\tilde{m}$ is called a fuzzifying measure on fuzzifying measurable space $(X, \tilde{\mathfrak{R}})$, and $(X, \tilde{\mathfrak{R}}, \tilde{m})$ is called a fuzzifying measure space. If $\tilde{m}$ only satisfy (1), (2), (3), then is called a lower-continuity fuzzifying measure. If $\tilde{m}$ only satisfy (1), (2), (3'), then is called an upper-continuity fuzzifying measure.

2. The Structure of Fuzzifying Measure

Definition 2.1. Let $\tilde{m}$ is a fuzzifying measure defined on fuzzifying measure space $(X, \tilde{\mathfrak{R}}, \tilde{m})$, then $\tilde{m}$ is considered having fuzzifying null-additive property (denoted briefly as 0-add) /or fuzzifying null-subtractive property [9] (denoted briefly as 0-sub) if for any $A, B \in P(X)$:

$$\vdash (\forall A)(\forall B)(A \in \tilde{\mathfrak{R}} \wedge B \in \tilde{\mathfrak{R}} \wedge (\neg(B \in \tilde{m}))$$

$$\rightarrow ((A \cup B) \in \tilde{m} \leftrightarrow A \in \tilde{m})); \text{ /or}$$

$$\vdash (\forall A)(\forall B)(A \in \tilde{\mathfrak{R}} \wedge B \in \tilde{\mathfrak{R}} \wedge (\neg(B \in \tilde{m}))$$

$$\rightarrow ((A \cap B^c) \in \tilde{m} \leftrightarrow A \in \tilde{m})).$$

Theorem 2.1. Let $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{R}})$, then the following conclusions are equivalence.

$$1) \tilde{m} \text{ is fuzzifying null-additive;}$$

$$2) \vdash (\forall A)(\forall B)(A \in \tilde{\mathfrak{R}} \wedge B \in \tilde{\mathfrak{R}} \wedge (A \cap B = \emptyset) \wedge (\neg(B \in \tilde{m})) \rightarrow ((A \cup B) \in \tilde{m} \leftrightarrow A \in \tilde{m}));$$

$$3) \vdash (\forall A)(\forall B)(A \in \tilde{\mathfrak{R}} \wedge B \in \tilde{\mathfrak{R}} \wedge (B \subset A) \wedge (\neg(B \in \tilde{m})) \rightarrow ((A - B) \in \tilde{m} \leftrightarrow A \in \tilde{m})).$$

$$\text{Proof. (1)} \Rightarrow \text{(2) is obvious;}$$

$$\text{(2)} \Rightarrow \text{(3): If } B \subset A, A = (A - B) \cup B \text{ and } (A - B) \cap B = \emptyset \text{ then:}$$

$$\tilde{m}((A - B) \cup B) \beta \tilde{m}(A - B)$$

$$\geq \tilde{\mathfrak{R}}(A - B) \wedge \tilde{\mathfrak{R}}(B) \wedge (\tilde{m}(B) \alpha 0)$$

$$\tilde{m}(A) \beta \tilde{m}(A - B) \geq$$

$$\tilde{\mathfrak{R}}(A) \wedge \tilde{\mathfrak{R}}(B) \wedge (B \subset A) \wedge (\tilde{m}(B) \alpha 0);$$

$$\text{(3)} \Rightarrow \text{(1):}$$

$$\tilde{m}(A \cup B - B) \beta \tilde{m}(A \cup B) \geq \tilde{\mathfrak{R}}(A \cup B)$$

$$\wedge \tilde{\mathfrak{R}}(B) \wedge (B \subset A \cup B) \wedge (\tilde{m}(B) \alpha 0)$$

$$\text{So } \tilde{m}(A) \beta \tilde{m}(A \cup B) \geq \tilde{\mathfrak{R}}(A \cup B) \wedge \tilde{\mathfrak{R}}(B)$$

$$\wedge (\tilde{m}(B)\alpha 0) \geq \tilde{\mathfrak{R}}(A) \wedge \tilde{\mathfrak{R}}(B) \wedge (\tilde{m}(B)\alpha 0).$$

Theorem 2.2. Let $\tilde{m}$ is an upper-continuity fuzzifying measure on $(X, \tilde{\mathfrak{R}}_1)$ and $\tilde{m}$ is 0-add $(\tilde{\mathfrak{R}}_1 = \{A | \tilde{\mathfrak{R}}(A) \geq 1, A \in P(X)\})$, for any $A \in \tilde{\mathfrak{R}}_1$, $\{B_n, n \in N\} \subseteq \tilde{\mathfrak{R}}_1, \forall n, B_{n+1} \subseteq B_n$. We have $\vdash (\forall A)(\forall B_n)((\neg(\lim(B_n \in \tilde{m}))) \rightarrow (\lim((A \cup B_n) \in \tilde{m}) \leftrightarrow A \in \tilde{m}))$.

Proof. $\tilde{m}$ is defined on $(X, \tilde{\mathfrak{R}}_1)$ so $\tilde{m}(\bigcap_{n=1}^{\infty} B_n) = \lim \tilde{m}(B_n)$. Let $B = \bigcap_{n=1}^{\infty} B_n$, and for $\{A \cup B_n, n \in N\}$, $A \cup B_{n+1} \subseteq A \cup B_n$, so $\tilde{m}(\bigcap_{n=1}^{\infty} (A \cup B_n)) = \lim \tilde{m}(A \cup B_n)$.

$$\begin{aligned} \text{Then } \lim \tilde{m}(A \cup B_n) \beta \tilde{m}(A) &= \\ \tilde{m}(\bigcap_{n=1}^{\infty} (A \cup B_n)) \beta \tilde{m}(A) &= \tilde{m}(A \cup B) \beta \tilde{m}(A) \\ \geq \tilde{m}(B)\alpha 0 &= \lim \tilde{m}(B_n)\alpha 0. \end{aligned}$$

Theorem 2.3. Let $\tilde{m}$ is a lower-continuity fuzzifying measure on $(X, \tilde{\mathfrak{R}}_1)$ and $\tilde{m}$ is 0-sub, for any $A \in \tilde{\mathfrak{R}}_1$, $\{B_n, n \in N\} \subseteq \tilde{\mathfrak{R}}_1, \forall n, B_{n+1} \subseteq B_n$. We have

$$\begin{aligned} \vdash (\forall A)(\forall B_n)((\neg(\lim(B_n \in \tilde{m}))) \rightarrow \\ (\lim((A \cap B_n^c) \in \tilde{m}) \leftrightarrow A \in \tilde{m})). \end{aligned}$$

Proof. Let $B = \bigcap_{n=1}^{\infty} B_n$, $\tilde{m}$ is defined on $(X, \tilde{\mathfrak{R}}_1)$, so $\tilde{m}(\bigcap_{n=1}^{\infty} B_n) = \lim \tilde{m}(B_n)$.

For $\{A \cap B_n^c, n \in N\}$, we have $A \cap B_n^c \subseteq A \cap B_{n+1}^c$, $\tilde{m}(\bigcup_{n=1}^{\infty} (A \cap B_n^c)) = \lim \tilde{m}(A \cap B_n^c)$.

$$\begin{aligned} \text{Then} \\ \lim \tilde{m}(A \cap B_n^c) \beta \tilde{m}(A) &= \\ \tilde{m}(\bigcup_{n=1}^{\infty} (A \cap B_n^c)) \beta \tilde{m}(A) &= \tilde{m}(A \cap B^c) \beta \tilde{m}(A) \\ \geq \tilde{m}(B)\alpha 0 &= \lim \tilde{m}(B_n)\alpha 0. \end{aligned}$$

Now, we want to point out that there has any fuzzifying measure ($\tilde{m}$), which has not fuzzifying null-additive property.

Example 2.1. Let $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{R}}_1)$, $\tilde{\mathfrak{R}}_1 = 2X, X = \{a, b\}$,

$$\begin{aligned} \tilde{m}(A) &= \begin{cases} 1, A = X \\ 0.3, A \neq X \end{cases} \quad \text{If } A = \{a\}, B = \{b\}, \\ \tilde{m}(A \cup B) \beta \tilde{m}(A) &= 1 \beta 0.3 = 0.3, \text{ but} \\ \tilde{m}(B)\alpha 0 &= 0.3 \alpha 0 = 0.7. \end{aligned}$$

So $\tilde{m}$ has not fuzzifying null-additive property.

Definition 2.2. Let $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{R}})$ and satisfies:

For any $A, B_n \in P(X), n \in N$

$$\begin{aligned} 1) \vdash (\forall A)(\forall B_n)(A \in \tilde{\mathfrak{R}} \wedge B_n \in \tilde{\mathfrak{R}} \wedge (\neg(\lim(B_n \in \tilde{m}))) \\ \rightarrow (\lim((A \cup B_n) \in \tilde{m}) \leftrightarrow A \in \tilde{m})); \\ 1') \vdash (\forall A)(\forall B_n)(A \in \tilde{\mathfrak{R}} \wedge B_n \in \tilde{\mathfrak{R}} \wedge (\neg(\lim(B_n \in \tilde{m}))) \\ \rightarrow (\lim((A \cap B_n^c) \in \tilde{m}) \leftrightarrow A \in \tilde{m})). \end{aligned}$$

Then $\tilde{m}$ is considered having fuzzifying auto-continuity property (denoted briefly as Fautoc.). If $\tilde{m}$ only satisfies (1) (or (1')), $\tilde{m}$ is considered having fuzzifying upper-auto (or lower-auto) continuity property (denoted briefly as Fautoc. $\downarrow$ (or Fautoc. $\uparrow$)) [11, 12].

Theorem 2.4. Let $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{R}})$, then

- 1) If $\tilde{m}$ is Fautoc. $\downarrow$, $\tilde{m}$ is 0-add.
- 2) If $\tilde{m}$ is Fautoc. $\uparrow$, $\tilde{m}$ is 0-sub.

Proof. (1) For any $A, B \in P(X)$, let $B_n = B$, $\forall n \in N$, then $A \cup B_n = A \cup B$.

$$\begin{aligned} \text{So} \\ \lim \tilde{m}(A \cup B_n) &= \sup_{n \in N} \inf_{m \geq n} \tilde{m}(A \cup B_n) = \tilde{m}(A \cup B); \\ \lim \tilde{m}(B_n) &= \sup_{n \in N} \inf_{m \geq n} \tilde{m}(B_n) = \tilde{m}(B) \end{aligned}$$

$$\begin{aligned} \text{And} \\ \tilde{m}(A \cup B) \beta \tilde{m}(A) &= \lim \tilde{m}(A \cup B_n) \beta \tilde{m}(A) \\ \geq \tilde{\mathfrak{R}}(A) \wedge (\prod_{n \in N} \tilde{\mathfrak{R}}(B_n)) \wedge (\lim \tilde{m}(B_n)\alpha 0) \end{aligned}$$

$= \tilde{\mathfrak{R}}(A) \wedge \tilde{\mathfrak{R}}(B) \wedge (\tilde{m}(B)\alpha 0)$. Similarly we can prove (2).

Example 2.2. $\tilde{m}$ is a fuzzifying measure on $(X, \tilde{\mathfrak{R}}_1)$, let $\tilde{\mathfrak{R}}_1 = 2X, X = \{1, 2, 3, \dots\}$

$$\tilde{m}(A) = \begin{cases} (2/\pi) \arctg(\text{car } A \sum_{i \in A} (1/2^i)), A \neq X \\ 1, A = X \end{cases}$$

(car A = the number of the members in set A).

Let $B_n = \{n, n+1, n+2, \dots\}$, then is easy to check: $\lim \tilde{m}(A \cap B_n^c) \beta \tilde{m}(A) = 1, \lim \tilde{m}(B_n)\alpha 0 = 1$.

So $\tilde{m}$ is Fautoc. $\uparrow$.

But $\lim \tilde{m}(A \cup B_n) \beta \tilde{m}(A) = 0 \beta 1 = 0$, $\tilde{m}$ is not Fautoc. $\downarrow$.

Proposition 2.1. If $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{R}})$, and X is finite, then $\tilde{m}$ is Fautoc. $\uparrow$ if and only if $\tilde{m}$ is Fautoc. $\downarrow$.

Definition 2.3. Let $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{R}})$, $\tilde{m}$ is fuzzifying uniformly upper-auto continuous (denoted briefly as Fu.autoc. $\downarrow$) if it satisfies: for any $\varepsilon > 0$, exist

$\delta = \delta(\varepsilon) > 0$, such that (1)

$$\begin{aligned} & \vdash (\forall A)(\forall B)(A \in \tilde{\mathfrak{X}} \wedge B \in \tilde{\mathfrak{X}} \wedge (B \in \tilde{m} \rightarrow \delta)) \\ & \rightarrow (A \cup B \in \tilde{m} \rightarrow G_3^+(A, \varepsilon)) \wedge (G_3^-(A, \varepsilon) \rightarrow \\ & A \cup B \in \tilde{m})) ; \end{aligned}$$

(Let $[G_3^+(A, \varepsilon)] = \min(1, \tilde{m}(A) + \varepsilon)$,

$$[G_3^-(A, \varepsilon)] = \max(0, \tilde{m}(A) - \varepsilon) .$$

$\tilde{m}$ is fuzzifying uniformly lower-auto continuous (denoted briefly as Fu.autoc. $\uparrow$) if it satisfies: for any $\varepsilon > 0$, exist $\delta = \delta(\varepsilon) > 0$, such that (1')

$$\begin{aligned} & \vdash (\forall A)(\forall B)(A \in \tilde{\mathfrak{X}} \wedge B \in \tilde{\mathfrak{X}} \wedge (B \in \tilde{m} \rightarrow \delta)) \\ & \rightarrow (A \cap B^c \in \tilde{m} \rightarrow G_3^+(A, \varepsilon)) \wedge (G_3^-(A, \varepsilon) \rightarrow \\ & A \cap B^c \in \tilde{m})) ; \end{aligned}$$

If $\tilde{m}$ satisfies (1) and (1'), then $\tilde{m}$ is fuzzifying uniformly auto continuous (denoted briefly as Fu.autoc.).

Theorem 2.5. Let $\tilde{m}$ is a fuzzifying measure on $(X, \tilde{\mathfrak{X}}_1)$, and then $\tilde{m}$ is Fu.autoc. $\downarrow$ if and only if $\tilde{m}$ is Fu.autoc. $\uparrow$. (We must point out that if $\tilde{m}$ is defined on $(X, \tilde{\mathfrak{X}})$, such proposition is not value).

Proof. $\Rightarrow$: For any $\varepsilon > 0$, exist $\delta = \delta(\varepsilon) > 0$, $A, B \in P(X)$. By $A = (A \cap B^c) \cup (A \cap B)$, and $\tilde{m}$ is defined on $(X, \tilde{\mathfrak{X}}_1)$. So

$$\tilde{m}(A \cap B) \leq \tilde{m}(B) ; \tilde{m}(A \cap B)\alpha\delta \geq \tilde{m}(B)\alpha\delta .$$

By $\tilde{m}$ is Fu.autoc. $\downarrow$, we have:

$$\begin{aligned} & \tilde{m}((A \cap B^c) \cup (A \cap B))\alpha \\ & [G_3^+((A \cap B^c), \varepsilon)] \geq \tilde{m}(A \cap B)\alpha\delta \geq \tilde{m}(B)\alpha\delta . \end{aligned}$$

$$\begin{aligned} & \text{So } \min(1, 1 - \tilde{m}((A \cap B^c) \cup (A \cap B))) \\ & + \min(1, \tilde{m}(A \cap B^c) + \varepsilon) \geq \tilde{m}(B)\alpha\delta \\ & \min(1, 1 - \tilde{m}(A) + \min(1, \tilde{m}(A \cap B^c) + \varepsilon)) \\ & \geq \tilde{m}(B)\alpha\delta \\ & \min(1, 1 - \max(0, \tilde{m}(A) - \varepsilon) + \tilde{m}(A \cap B^c)) \\ & \geq \tilde{m}(B)\alpha\delta . \end{aligned}$$

Therefore

$$[G_3^-(A, \varepsilon)]\alpha\tilde{m}(A \cap B^c) \geq \tilde{m}(B)\alpha\delta .$$

Similarly

$$\begin{aligned} & [G_3^-((A \cap B^c), \varepsilon)]\alpha\tilde{m}((A \cap B^c) \cup (A \cap B)) \\ & \geq \tilde{m}(A \cap B)\alpha\delta \geq \tilde{m}(B)\alpha\delta . \end{aligned}$$

Therefore

$$\begin{aligned} & \min(1, 1 - \max(0, \tilde{m}(A \cap B^c) - \varepsilon) + \tilde{m}(A)) \\ & \geq \tilde{m}(B)\alpha\delta \\ & \min(1, 1 - \tilde{m}(A \cap B^c) + \tilde{m}(A) + \varepsilon) \geq \tilde{m}(B)\alpha\delta \\ & \min(1, 1 - \tilde{m}(A \cap B^c) + \min(1, \tilde{m}(A) + \varepsilon)) \\ & \geq \tilde{m}(B)\alpha\delta . \end{aligned}$$

$$\text{So } \tilde{m}(A \cap B^c)\alpha[G_3^+(A, \varepsilon)] \geq \tilde{m}(B)\alpha\delta$$

$$([G_3^-(A, \varepsilon)]\alpha\tilde{m}(A \cap B^c))$$

$$\wedge (\tilde{m}(A \cap B^c)\alpha[G_3^+(A, \varepsilon)]) \geq \tilde{m}(B)\alpha\delta .$$

$\tilde{m}$ is Fu.autoc. $\uparrow$.

$\Leftarrow$: Using the same method, we can prove the part of result easily.

Theorem 2.6. Let $\tilde{m}$ is a fuzzifying measure defined on $(X, \tilde{\mathfrak{X}})$, then:

1) If $\tilde{m}$ is Fu.autoc. $\downarrow$, $\tilde{m}$ is Fautoc. $\downarrow$;

2) If $\tilde{m}$ is Fu.autoc. $\uparrow$, $\tilde{m}$ is Fautoc. $\uparrow$.

Proof. (1) For any $A, B \in P(X)$, $\{B_n \mid n \in N\} \subseteq P(X)$, $\tilde{m}$ is Fu.autoc. $\downarrow$, so for any $\varepsilon > 0$, exist $\delta = \delta(\varepsilon) > 0$, and

$$\begin{aligned} & (\tilde{m}(A \cup B)\alpha[G_3^+(A, \varepsilon)]) \wedge ([G_3^-(A, \varepsilon)]\alpha\tilde{m}(A \cup B)) \\ & \geq \tilde{\mathfrak{X}}(A) \wedge \tilde{\mathfrak{X}}(B) \wedge (\tilde{m}(B)\alpha 0) . \end{aligned}$$

For such ε , exist $N > 0$, if $n > N$, we have

$$\begin{aligned} & (\tilde{m}(A \cup B_n)\alpha[G_3^+(A, \varepsilon)]) \wedge ([G_3^-(A, \varepsilon)]\alpha\tilde{m}(A \cup B_n)) \\ & \geq \tilde{\mathfrak{X}}(A) \wedge \tilde{\mathfrak{X}}(B) \wedge (\tilde{m}(B_n)\alpha\delta) \\ & \geq \tilde{\mathfrak{X}}(A) \wedge \tilde{\mathfrak{X}}(B_n) \wedge (\tilde{m}(B_n)\alpha 0) . \end{aligned}$$

$$\text{So } \lim \tilde{m}(A \cup B_n)\beta\tilde{m}(A) \geq$$

$$\tilde{\mathfrak{X}}(A) \wedge \prod_{n \in N} \tilde{\mathfrak{X}}(B_n) \wedge (\lim \tilde{m}(B_n)\alpha 0) .$$

The second conclusion can be proved by the same method.

3. On the Product of Two Fuzzifying Measure

Definition 3.1. Let $\tilde{m}_1$ and $\tilde{m}_2$ are all fuzzifying measures defined on $(X, \tilde{\mathfrak{X}})$, we denote the product of $\tilde{m}_1$ and $\tilde{m}_2$ as $\tilde{m}_1\tilde{m}_2$ which is defined as: for any $A \in \tilde{\mathfrak{X}}$, $\tilde{m}_1\tilde{m}_2(A) = \tilde{m}_1(A) \wedge \tilde{m}_2(A)$ [16].

Now, we must prove that $\tilde{m}_1\tilde{m}_2$ is fuzzifying measures on $(X, \tilde{\mathfrak{X}})$, for any $A, B \in P(X)$:

$$1) \tilde{m}_1\tilde{m}_2(\emptyset) = \tilde{m}_1(\emptyset) \wedge \tilde{m}_2(\emptyset) = 0 ,$$

$$\tilde{m}_1\tilde{m}_2(X) = \tilde{m}_1(X) \wedge \tilde{m}_2(X) = 1 ;$$

$$2) \tilde{m}_1\tilde{m}_2(A) \alpha\tilde{m}_1\tilde{m}_2(B) = (\tilde{m}_1(A) \wedge \tilde{m}_2(A))$$

$$\alpha(\tilde{m}_1(B) \wedge \tilde{m}_2(B))$$

$$\geq (\tilde{m}_1(A)\alpha\tilde{m}_1(B)) \wedge (\tilde{m}_2(A)\alpha\tilde{m}_2(B))$$

$$\geq \tilde{\mathfrak{X}}(A) \wedge \tilde{\mathfrak{X}}(B) \wedge (A \subset B) ;$$

3) Using the same method we can prove $\tilde{m}_1\tilde{m}_2$ satisfies the conditions (3) and (3') of the Definition 1.4.

Definition 3.2. Let $\tilde{m}_1$ and $\tilde{m}_2$ are all fuzzifying

measures defined on $(X, \mathfrak{F})$, $\tilde{m}_1$ is said to be fuzzifying weakly absolutely continuous with respect to $\tilde{m}_2$ and is denoted by $\tilde{m}_1 \ll_{\omega} \tilde{m}_2$ if for any $A \in P(X)$:

$$\vdash (\forall A)(A \in \mathfrak{F} \rightarrow (\neg(A \in \tilde{m}_2) \rightarrow \neg(A \in \tilde{m}_1))) ;$$

$\tilde{m}_1$ is said to be strongly absolutely continuous with respect to $\tilde{m}_2$ and is denoted by $\tilde{m}_1 \ll_s \tilde{m}_2$ if for any $\varepsilon > 0$, exist $\delta = \delta(\varepsilon) > 0$, such that for any $A \in P(X)$:

$$\vdash (\forall A)(A \in \mathfrak{F} \rightarrow ((A \in \tilde{m}_2 \rightarrow \delta) \rightarrow (A \in \tilde{m}_1 \rightarrow \varepsilon))) .$$

Definition 3.3. $\tilde{m}$ is called to have property (s) (or property (s')) if for any $\{B_n\} \subset P(X)$, $A \in P(X)$:

$$\vdash (\forall n)((B_n \in \mathfrak{F}) \wedge (\neg(\lim(B_n \in \tilde{m})))$$

$$\rightarrow (\exists n_i)(\neg((\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} B_{n_i}) \in \tilde{m}))$$

(Or $\vdash (\forall n)((B_n \in \mathfrak{F}) \wedge (A \in \mathfrak{F}) \wedge (\neg(\lim(B_n \in \tilde{m}))) \rightarrow$

$$(\exists n_i)((A \cap (\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} B_{n_i})^c) \in \tilde{m} \leftrightarrow (A \in \tilde{m})) .$$

Theorem 3.1. 1) $\tilde{m}$ is a fuzzifying measure on $(X, \mathfrak{F}_1)$ and is Fautoc. $\downarrow$ if and only if $\tilde{m}$ is 0-add and has property (s);

2) $\tilde{m}$ is a fuzzifying measure on $(X, \mathfrak{F}_1)$ and is Fautoc. $\uparrow$ if and only if $\tilde{m}$ is 0-sub and has property (s').

Proof. We only prove the conclusion (1). $\Rightarrow$: We only need to prove that $\tilde{m}$ has property (s) because $\tilde{m}$ is 0-add has been proved by Theorem 2.4. For any $\{B_n\} \in P(X)$, $\forall \varepsilon > 0$, exist N_1 , such that:

$$\vdash \neg(\lim(B_n \in \tilde{m})) \rightarrow (B_{N_1} \in \tilde{m} \rightarrow \varepsilon/2) \dots (*) .$$

For B_{N_1} , by $\tilde{m}$ is Fautoc. $\downarrow$, we have

$$\vdash (\forall n_2)((n_2 > N_1) \wedge (\neg(\lim(B_{n_2} \in \tilde{m}))) \rightarrow$$

$$(\lim((B_{N_1} \cup B_{N_2}) \in \tilde{m}) \rightarrow B_{N_1} \in \tilde{m}) .$$

So exist $N_2 > N_1$ and

$$\vdash (\forall n_2)((n_2 > N_1) \wedge (\neg(\lim(B_{n_2} \in \tilde{m}))) \rightarrow$$

$$(((B_{N_1} \cup B_{N_2}) \in \tilde{m}) \rightarrow \min(1, \tilde{m}(B_{N_1}) + \varepsilon/4))) .$$

For equation (*), select some N_1 , we have

$$\vdash (\forall n_2)((n_2 > N_1) \wedge (\neg(\lim(B_{n_2} \in \tilde{m})))$$

$$\rightarrow (((B_{N_1} \cup B_{N_2}) \in \tilde{m}) \rightarrow \min(1, 3\varepsilon/4))) .$$

Do like this again and again we can find out such conclusion:

$$\vdash \neg(\lim(B_n \in \tilde{m})) \rightarrow$$

$$(\exists n_i)((\bigcup_{i=1}^{\infty} B_{n_i}) \in \tilde{m} \rightarrow \min(1, \varepsilon)) .$$

If $\varepsilon = 1$

$$\vdash \neg(\lim(B_n \in \tilde{m})) \rightarrow$$

$$(\exists n_{i(1)})(\bigcup_{i=1}^{\infty} B_{n_{i(1)}} \in \tilde{m} \rightarrow 1) .$$

If $\varepsilon = 1/2$

$$\vdash \neg(\lim(B_n \in \tilde{m})) \rightarrow$$

$$(\exists n_{i(2)})(\bigcup_{i=1}^{\infty} B_{n_{i(2)}} \in \tilde{m} \rightarrow 1/2) .$$

Generally, if $\varepsilon = 1/j$

$$\vdash \neg(\lim(B_n \in \tilde{m})) \rightarrow$$

$$(\exists n_{i(j)})(\bigcup_{i=1}^{\infty} B_{n_{i(j)}} \in \tilde{m} \rightarrow 1/j) , j=1, 2, 3 \dots$$

Therefore:

$$\vdash \neg(\lim(B_n \in \tilde{m})) \rightarrow (\exists n_i)(\neg((\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} B_{n_i}) \in \tilde{m})) .$$

$\Leftarrow$: For any $A \in P(X)$, $\{B_n\} \in P(X)$, exist

$\{B_{n_i}\} \subset \{B_n\}$, such that

$$(\prod_{n \in N} (\sum_{m \geq n} \tilde{m}(A \cup B_n))) \alpha \tilde{m}(A) =$$

$$\lim \tilde{m}(A \cup B_{n_i}) \alpha \tilde{m}(A) .$$

For $\{B_{n_i}\}$, by $\tilde{m}$ has property (s), we can find

$\{B_{n_{i_k}}\} \subset \{B_{n_i}\}$, then

$$\tilde{m}(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} B_{n_{i_k}}) \alpha 0 \geq \lim \tilde{m}(B_{n_{i_k}}) \alpha 0 . \text{ By } \tilde{m} \text{ is 0-}$$

add, so $\tilde{m}(A \cup (\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} B_{n_{i_k}})) \beta \tilde{m}(A)$

$$\geq \tilde{m}(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} B_{n_{i_k}}) \alpha 0 \geq \lim \tilde{m}(B_{n_{i_k}}) \alpha 0 .$$

And $(\inf_{m \in N} \sup_{n \geq m} \tilde{m}(A \cup B_n)) \beta \tilde{m}(A)$

$$= \lim \tilde{m}(A \cup B_{n_i}) \alpha \tilde{m}(A)$$

$$\geq \tilde{m}(A \cup (\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} B_{n_{i_k}})) \alpha \tilde{m}(A) . \text{ So}$$

$$(\inf_{m \in N} \sup_{n \geq m} \tilde{m}(A \cup B_n)) \beta \tilde{m}(A) \geq \lim \tilde{m}(B_{n_{i_k}}) \alpha 0 .$$

$$\text{Then } (\sup_{m \in N} \inf_{n \geq m} \tilde{m}(A \cup B_n)) \beta \tilde{m}(A) \geq \lim \tilde{m}(B_n) \alpha 0 ,$$

$$\lim \tilde{m}(A \cup B_n) \beta \tilde{m}(A) \geq \lim \tilde{m}(B_n) \alpha 0 .$$

Theorem 3.2. $\tilde{m}_1$ and $\tilde{m}_2$ are fuzzifying measures defined on $(X, \mathfrak{F}_1)$, and $\tilde{m}_1 \ll_{\omega} \tilde{m}_2$.

1) If $\tilde{m}_1$ and $\tilde{m}_2$ are all 0-add, then so is $\tilde{m}_1 \tilde{m}_2$;

2) If $\tilde{m}_1$ and $\tilde{m}_2$ are all Fautoc., then so is $\tilde{m}_1 \tilde{m}_2$.

Proof. (1) For any $A, B \in P(X)$,

$$\tilde{m}_1 \tilde{m}_2 (A \cup B) \beta \tilde{m}_1 \tilde{m}_2 (A)$$

$$= (\tilde{m}_1(A \cup B) \wedge \tilde{m}_2(A \cup B))$$

$$\beta (\tilde{m}_1(A) \wedge \tilde{m}_2(A))$$

$$\geq (\tilde{m}_1(A \cup B) \beta \tilde{m}_1(A)) \wedge (\tilde{m}_2(A \cup B) \beta \tilde{m}_2(A))$$

$$\geq (\tilde{m}_1(B)\alpha 0) \wedge (\tilde{m}_2(B)\alpha 0).$$

By $\tilde{m}_1 \ll_{\omega} \tilde{m}_2$, we have

$$(\tilde{m}_1(B)\alpha 0) \geq (\tilde{m}_2(B)\alpha 0), \text{ then}$$

$$\tilde{m}_1(B) \leq \tilde{m}_2(B). \text{ So}$$

$$\tilde{m}_1 \tilde{m}_2(A \cup B) \beta \tilde{m}_1 \tilde{m}_2(A)$$

$$\begin{aligned} &\geq (\tilde{m}_1(A)\alpha 0) \wedge (\tilde{m}_2(B)\alpha 0) = \tilde{m}_2(B)\alpha 0 \\ &\geq (\tilde{m}_1(B) \wedge \tilde{m}_2(B))\alpha 0 = \tilde{m}_1 \tilde{m}_2(B)\alpha 0. \end{aligned}$$

2) We only prove if $\tilde{m}_1$ and $\tilde{m}_2$ are all Fautoc. $\downarrow$, then so is $\tilde{m}_1 \tilde{m}_2$. It is obvious that $\tilde{m}_1 \tilde{m}_2$ is 0-add, by Theorem 3.1; we only need to prove that $\tilde{m}_1 \tilde{m}_2$ has property (s).

For $\{B_n\} \subset P(X)$, $n=1, 2, 3, \dots$ $\tilde{m}_1$ has property (s), so exist $\{B_{n_i}\} \subset \{B_n\}$, we have

$$\tilde{m}_1(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} B_{n_i})\alpha 0 \geq \lim \tilde{m}_1(B)\alpha 0. \text{ For } \{B_{n_i}\},$$

by $\tilde{m}_2$ has property (s), exist $\{B_{n_{ij}}\} \subset \{B_{n_i}\}$

$$\tilde{m}_2(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}})\alpha 0 \geq \lim \tilde{m}_2(B_n)\alpha 0. \text{ So}$$

$$\tilde{m}_1(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}})\alpha 0 \geq \tilde{m}_1(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} B_{n_i})\alpha 0$$

$$\geq \lim \tilde{m}_1(B_n)\alpha 0. \text{ Then}$$

$$\begin{aligned} &(\tilde{m}_1(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}})\alpha 0) \wedge (\tilde{m}_2(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}})\alpha 0) \\ &\geq (\lim \tilde{m}_1(B_n)\alpha 0) \wedge (\lim \tilde{m}_2(B_n)\alpha 0). \end{aligned}$$

$$(\tilde{m}_1(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}}) \wedge \tilde{m}_2(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}}))\alpha 0$$

$$\geq (\lim \tilde{m}_1(B_n)\alpha 0) \wedge (\lim \tilde{m}_2(B_n)\alpha 0)$$

$$\tilde{m}_1 \tilde{m}_2(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}})\alpha 0 \geq (\lim \tilde{m}_1(B_n)\alpha 0)$$

$$\wedge (\lim \tilde{m}_2(B_n)\alpha 0).$$

By $\tilde{m}_1 \ll_{\omega} \tilde{m}_2$, we have

$$\lim \tilde{m}_1(B_n)\alpha 0 \geq \lim \tilde{m}_2(B_n)\alpha 0 \text{ and}$$

$$\lim \tilde{m}_1(B_n) \leq \lim \tilde{m}_2(B_n).$$

$$\text{So } \tilde{m}_1 \tilde{m}_2(\bigcap_{t=1}^{\infty} \bigcup_{j=t}^{\infty} B_{n_{ij}})\alpha 0 \geq \lim \tilde{m}_2(B_n)\alpha 0$$

$$= (\lim \tilde{m}_1(B_n) \wedge \lim \tilde{m}_2(B_n))\alpha 0$$

$$= \lim \tilde{m}_1 \tilde{m}_2(B_n)\alpha 0. \text{ Then } \tilde{m}_1 \tilde{m}_2 \text{ has property (s).}$$

4. Conclusions

Used a sematic method of continuous-valued logic system, fuzzifying measurable theory that generalizes the classical measurable theory is

established. In this article, the structure of the new theory has been studied and some satisfaction results have been gained. We also study the product of two fuzzifying measures and consider a problem, which is like the third of the open problems in fuzzy measure presented by Z. Wang. We have solved this problem satisfactorily in the new theory.

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