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Singular Perturbation Margin and Generalized Gain Margin for Linear Periodically Time Varying Systems

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Abstract: Singular Perturbation Margin (SPM) and Generalized Gain Margin (GGM) are Phase Margin (PM) and Gain Margin (GM) like stability metrics. In this paper, the problem of SPM and GGM assessment for Linear Periodically Time-Varying (LPTV) systems is formulated, which provides a necessary step for the development of the theoretically based and practically efficient SPM and GGM analysis methodology for multivariable Nonlinear Time-Varying (NLTV) systems. Here, Chang transformation makes it possible to reduce the SPM analysis for Hill equations, which is essentially a stability problem of higher order LPTV systems due to the SPM gauge introduced dynamics, to second order LPTV systems. Based upon Floquet Theory, when different types of LPTV SPM gauges are designed, the SPM and GGM assessment methods for the second order and general order LPTV systems are established and stated by the corresponding lemmas, theorems and corollaries. The effectiveness of such methods are illustrated by an examples of a flexible inverted pendulum.

Keywords: Stability margin, Singular perturbation, Regular perturbation, Linear periodically time-varying systems, Hill equation.

1. Introduction

Research on periodically time-varying systems can be traced back to M. Faraday [1] in 1830s; the first detailed theory study for Linear Periodically Time-Varying (LPTV) systems is given by E. Mathieu [2]; G. Floquet established Floquet theory [3] in 1883, which forms the basis of a great many of the descriptions of parametric behaviors [4]; one of the most significant earlier papers on periodically time-varying systems was G. W. Hill [5], whose name has been given to the general class of the second order periodic differential equations, and his research is the first investigation of a practical problem in such field. Due to a growing number of applications of LPTV systems in different fields, since the first half of the 20^{th} century, more study effort was put on Meissner equation [6], Mathieu equation [7], the solutions [8] and the stability behavior of the above equations [9], etc. Through the continuous hard work of scholars for the recent 60-70 years, the research methodology of LPTV systems has been formed a relatively mature body of knowledge, and can be roughly divided into three categories: 1, find the time invariant equivalent system for the periodically time-varying system and further discuss the problems of stability and pole assignment [10-13]; 2, analyze the controllability and observability of LPTV systems based on the state transition matrix of one period [14,15]; 3, Lyapunov methods [16-19].

Stability is a key problem in the design of automatic control systems, and in engineering applications, not only the question whether the system is stable is must be known information, but also the quantitative criteria of stability is pivotally demanded. Recently, Singular Perturbation Margin (SPM) and Generalized Gain Margin (GGM) [20] have been proposed as the classical Phase Margin (PM) and Gain Margin (GM) like stability metrics for Single Input Single Output (SISO) Linear Time Invariant (LTI) and Nonlinear Time Invariant (NLTI) systems [21] from the view of the singular perturbation and the regular perturbation. Due to the distinguishing feature of the exponential stability of the equilibrium point, i.e. the robustness to the nonvanishing regular perturbations, the concepts of SPM and GGM have been investigated towards to the establishment of the quantitative exponential stability metrics for general Multi Input Multi Output (MIMO) Nonlinear Time Varying (NLTV) systems, defined by the maximum singular perturbation parameter, denoted by $\varepsilon_{\max}$, and the maximum and minimum regular perturbation parameters, denoted by $k_{\min}$ and $k_{\max}$ that render the perturbed closed-loop system at onset of instability to gauge the capability in accommodating singular perturbations (parasitic dynamics) and regular perturbations (parametric dispersions). Some basic results of stability margin are listed in [22]. Furthermore, the long term effort of such research is the development of the theoretically based and practically efficient SPM and GGM analysis methodology to facilitate the nonlinear and time-varying control system design.

Based upon Floquet Theory, which provides necessary and sufficient conditions of the exponential stability (equivalent to the asymptotical stability and uniformly asymptotical stability for LPTV systems), the present paper begins the aforementioned research road map of SPM and GGM analysis for LPTV systems. The assessment theorems of second order LPTV systems are individually conceived and special attention is paid to the most commonly occurring periodic differential equations, Hill Equations. From the theoretical point of view, the LPTV SPM and GGM assessment methods developed in the present paper could be directly carried out when applied to nonlinear periodically time varying systems at the null equilibrium by Jacobian linearization thereof according to the equilibrium theorems [21], and would be a necessary step to be further applied to stability margin analysis of many challenging nonlinear control problems when more nonlinear properties are considered, such as domain of attraction. From the practical point of view, the SPM and GGM have bijective relationships with the PM and GM [20], respectively, and the results here are potentially useful to facilitate the periodic system design due to the large range of application-oriented LPTV problems [4].

In this paper, the problem of SPM and GGM analysis for LPTV systems is formulated in Section Objectives; the main results: decoupling technique of the singularly perturbed LPTV systems; SPM and GGM gauge design; SPM and the GGM assessment methods, examples etc., are listed in Section 3, Section 4 and Section 5; Section 6 concludes the whole paper's with some insightful remarks.

The symbols used in this paper are shown in Table 1.

Table 1. Symbols.

Symbols	Meanings
ε	Singular Perturbation Parameter
$\varepsilon_{\max}$	Exact SPM of the Nominal System
$\hat{\varepsilon}_{\max}$	Estimated SPM of the Nominal System
k	Regular Perturbation Parameter
$k_{\min}, k_{\max}$	Exact GGM of the Nominal System
$\hat{k}_{\min}, \hat{k}_{\max}$	Estimated GGM of the Nominal System
G_{SPM}	SISO Linear SPM Gauge
G_{GGM}	SISO GGM Gauge
$\mathbb{R}$	Field of Real Numbers
Ω	A time interval in $\mathbb{R}$ which could be a finite interval $\left[t_0, t_f\right]$ or an infinite interval $\left[t_0, \infty\right)$
T	System Period
$\varphi(T,0)$	Discrete Transition Matrix (DTM)
$\varphi_{\mathrm{pert},k}(\pi,0)$	DTM of Regularly Perturbed System
$\varphi_{\mathrm{pert},\varepsilon}(\pi,0)$	DTM of Singularly Perturbed System
$W(t)$	Wronskian Matrix
λ_i	Floquet Characteristic Multiplier
$\rho_i(t)$	Parallel Differential Eigenvalue
$A_{slow,p}(t,\varepsilon)$	System Matrix of p^{th}-order approximated slow system
$A_{nom}(t)$	System Matrix of the Nominal System

2. Objectives

The SPM and GGM analysis for a closed-loop and well-designed LPTV control system (also called Nominal System)

$$\dot{x} = A_{nom}(t)x , \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $A_{nom}(t)$ is bounded and continuously differentiable, satisfying

$$A_{nom}(t) = A_{nom}(t+T) \tag{2}$$

with T the system period, is referred as to the quantitative assessment of the capability of the exponentially stable null equilibrium (origin) in accommodating singular perturbations and regular perturbations:

1) Design the SPM gauge and GGM gauge for the specific perturbations;

2) Determine the stability of the perturbed system using stability criterion when different singular perturbation parameters and gain parameters are given, respectively;

3) Obtain the SPM and GGM by the marginally stable situation of the perturbed system;

4) Reveal the relation between the characteristics of the nominal system (1) and its exact SPM and GGM or estimated SPM and GGM.

3. Stability Criterion and SPM/GGM Gauges

We first introduce a necessary and sufficient exponential stability condition for LPTV systems that will be used in the sequel.

Lemma 1. [4] *The natural response of an LPTV system is stable, if and only if no eigenvalues of the Discrete Transition Matrix (DTM) have magnitudes greater than one and that an eigenvalue with unity magnitude is not degenerate.*

The eigenvalues of the DTM is also called Floquet characteristic multipliers of the periodic system. The SPM and GGM analysis is based on the specific gauge design, i. e. gauge based stability metrics. The SPM gauge can be designed as either an LTI or a Linear Time Varying (LTV) system demonstrating the interested properties of the singular perturbation term, which is a priori for a specific control design, such as the simplification assumptions of the high fidelity mathematical model.

The *SISO LTV SPM Gauge* G_{SPM} is given by

$$\varepsilon \dot{z} = A_f(t)z + B_f u, \quad y = C_f z, \tag{3}$$

where $A_f(t)$ is a $m \times m$ bounded and continuously differentiable function matrix, B_f and C_f are bounded $m \times 1$ and $1 \times m$, constant matrices, u and y are the input and output of the SPM gauge. If the system matrix $A_f(t)$ is a periodic function matrix satisfying $A_f(t) = A_f(t+T)$ with T the system period, system (3) is called an *SISO LPTV SPM Gauge*; furthermore if the system matrix of the SISO LTV SPM Gauge satisfies the semi-proper condition (also known as the functional commutativity)

$$A_f(t)A_f(\tau) = A_f(\tau)A_f(t) \quad \text{for all } t, \tau \in \Omega, \tag{4}$$

where Ω is a time interval in $\mathbb{R}$ which could be a finite interval $\left[t_0, t_f\right]$ or an infinite interval $\left[t_0, \infty\right)$, system (3) is called a *Semi-Proper SISO LPTV SPM Gauge*; if the system matrix $A_f(t)$ satisfies the slowly varying condition [23], system (3) is called a *Slowly Varying SISO LPTV SPM Gauge*; if the system matrix $A_f(t)$ is a constant matrix, system (3) is called an *SISO LTI SPM Gauge,* which is a special case of the SISO LPTV SPM Gauge.

The *SISO LTV GGM Gauge, denoted by* G_{GGM}, is a scalar function of time *k(t)*, and the *SISO LTI GGM Gauge* is a constant *k* in the closed-loop system.

4. Perturbed Hill Equation

4.1. System Model and Perturbations

The transition matrix of a general order LPTV system is analytically intractable in Lemma 1, but for the second-order cases, the Floquet characteristic multipliers can be written as

$$\lambda_{1,2} = \frac{r}{2} \pm \sqrt{\left(\frac{r}{2}\right)^2 - 1} \quad \text{where } r = trace\{\varphi(T,0)\}$$

and $\varphi(T,0)$ is the Discrete Transition Matrix (DTM) of the second-order LPTV system. For the second-order cases, the general form

$$\ddot{x} + \alpha_1(t)\dot{x} + \alpha_0(t) = 0$$

can be converted into an expression without a first derivative, and the classical form is [4]

$$\ddot{y} + (a - 2q\psi(t))y = 0, \quad \psi(t) = \psi(t+\pi), \tag{5}$$

where function $\psi(t)$ has a period of π and a, q are constant parameters, is identified as *Hill equation*. Hill equation is not only the model in many applications, but also has significant meaning for general second-order equations. When the singular and regular perturbations are considered in the closed-loop, the block diagram of Hill equation with corresponding perturbation sites is shown by Fig. 1.

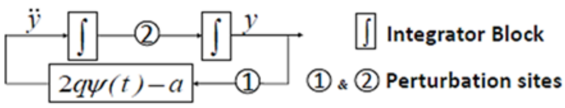

Fig. 1. Block Diagram of Hill Equation with Perturbations.

Design the SISO LTV SPM gauge G_{SPM} at the perturbation sites, which is essentially the fast system in the multi time scale closed loop, and the singularly perturbed system is given by

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)z \\ \varepsilon \dot{z} &= A_{21}(t)x + A_{22}(t)z, \end{aligned} \tag{6}$$

where the singular perturbation parameter ε describes the time-scale separation, $A_{11}(t), A_{12}(t), A_{21}(t), A_{22}(t)$ are function matrices that are dependent on the perturbation site and the SPM gauge design, and $z \in \mathbb{R}^m$ is state vector of the fast dynamics. When the SISO LTI SPM gauges is designed at the perturbation sites, which might have the equation forms as [20] minimum-phase gauge, zero-error gauge or approximately-linear phase gauge, etc., the singularly perturbed system is given by

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)z \\ \varepsilon\dot{z} &= A_{21}(t)x + A_{22}z \end{aligned} \tag{7}$$

where A_{22} becomes a constant matrix. When $\varepsilon \to 0$, the systems (6) and (7) are reduced from $(n+m)$-th to n-th order nominal system (1).

With the regular perturbation parameter $k \in (-\infty, \infty)$ the regularly perturbed system can be written as

$$\dot{x} = A(t,k)x \tag{8}$$

and when the regular perturbation parameter $k = 1$, the regularly perturbed model (8) becomes the nominal system, $A(t,1) = A_{nom}(t)$.

4.2. Decoupled Slow and Fast Systems

The singularly perturbed system (7) can be totally decoupled to slow and fast system by Chang transformation.

Lemma 2. *With the SISO LTI SPM Gauge, the decoupled slow system of the singularly perturbed Hill equation (7) is still a Hill equation with the same period to the nominal system.*

Proof. Consider the two perturbation positions, Site 1 and Site 2, in the loop of Fig. 1.

Site 1: With the SISO LTI SPM Gauge, the system matrices of the singularly perturbed system in form of (6) are given by

$$\begin{aligned} &A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{12}(t) = \begin{bmatrix} 0 & \cdots & 0 \\ & (-a+2q\psi(t))C_f & \end{bmatrix}, \\ &A_{21} = B_f \begin{bmatrix} 1 & 0 \end{bmatrix}, A_{22} = A_f, \end{aligned}$$

where A_f, B_f, C_f are the parameter matrices of the SISO SPM gauge. Chang transformation [23] is given by

$$\begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \begin{bmatrix} I - \varepsilon H(t)L(t) & -\varepsilon H(t) \\ L(t) & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = S(t) \begin{bmatrix} x \\ z \end{bmatrix},$$

where function matrices *L(t)* and *H(t)* satisfy Differential Riccati Equation (DRE) and Differential Sylvester Equation (DSE)

$$\begin{aligned} -\dot{L}(t) &= L(t)A_{11} + \frac{1}{\varepsilon}A_{21} - L(t)A_{12}(t)L(t) - \frac{1}{\varepsilon}A_{22}L(t) \\ -\dot{H}(t) &= A_{11}H(t) + A_{12}(t) - H(t)L(t)A_{12}(t) \\ &\quad - \frac{1}{\varepsilon}H(t)A_{22} - A_{12}(t)L(t)H(t) \end{aligned}$$

For Chang transformation, the iterative solutions of the DRE and DSE in form of

$$L(t) = \sum_{j=0}^{\infty} \varepsilon^j \bar{L}_j(t), \quad H(t) = \sum_{j=0}^{\infty} \varepsilon^j \bar{H}_j(t), \tag{9}$$

where

$$\begin{aligned} &\bar{L}_0 = A_{22}^{-1}A_{21}, \quad \bar{H}_0 = A_{12}^{-1}A_{22} \\ &\bar{L}_{k+1} = A_{22}^{-1}\left[\dot{\bar{L}}_k + \bar{L}_k\left(A_{11} - A_{12}\bar{L}_k\right)\right], k = 0,1,2,... \\ &\bar{H}_{k+1} = \left[-\dot{\bar{H}}_k + \bar{H}_k\bar{L}_kA_{12} + \left(A_{11} - A_{12}\bar{L}_k\right)\bar{H}_k\right]A_{22}^{-1} \\ &\qquad k = 0,1,2,... \end{aligned}$$

and p^{th}-order approximated slow system is given by

$$A_{slow,p}(t,\varepsilon) = A_{11}(t) - A_{12}(t)L_p(t,\varepsilon) \tag{10}$$

Then, the slow system is given by

$$\begin{aligned} A_{slow} &= A_{11} - A_{12}(t)L(t) \\ &= \begin{pmatrix} 0 & 1 \\ (a-2q\psi(t))C_fL_1(t) & (a-2q\psi(t))C_fL_2(t) \end{pmatrix}, \end{aligned}$$

where $L_1(t), L_2(t)$ are the column vectors of $L(t)$, and the second order equation becomes

$$\ddot{x} + \left(2q\psi(t) - a\right)C_fL_2(t)\dot{x} + \left(2q\psi(t) - a\right)C_fL_1(t)x = 0 \tag{11}$$

By transformation

$$x(t) = \exp\left\{-\frac{1}{2}\int_0^t \left(2q\psi(t) - a\right)C_fL_2 dt\right\} y(t)$$

Eq. (11) is transformed to

$$\begin{aligned} \ddot{y} = -\Big\{&\left(2q\psi(t) - a\right)C_fL_1(t) - q\dot{\psi}(t)C_fL_2(t) \\ &- \frac{1}{4}\left(2q\psi(t) - a\right)^2\left(C_fL_1(t)\right)^2\Big\} y \end{aligned}$$

which is a Hill equation with the same period to function $\psi(t)$.

Site 2: With the SISO LTI SPM Gauge, the system matrices of the singularly perturbed system become

$$\begin{aligned} &A_{11} = \begin{bmatrix} 0 & 0 \\ -a+2q\psi(t) & 0 \end{bmatrix}, A_{12}(t) = \begin{bmatrix} 0 & C_f & 0 \\ 0 & \cdots & 0 \end{bmatrix}, \\ &A_{21} = B_f \begin{bmatrix} 0 & 1 \end{bmatrix}, A_{22} = A_f, \end{aligned}$$

By Chang transformation, the slow system is given by

$$\begin{aligned} A_{slow}(t) &= A_{11}(t) - A_{12}L(t) \\ &= \begin{pmatrix} C_fL_1(t) & C_fL_2(t) \\ -(a-2q\psi(t)) & 0 \end{pmatrix} \end{aligned}$$

and the second order slow system becomes

$$\ddot{x} - \left(C_f L_1 + \tfrac{C_f \dot{L}_2}{C_f L_2}\right)\dot{x} + \left(C_f \dot{L}_1 - \tfrac{C_f \dot{L}_2 C_f L_1}{C_f L_2} + \left(2q\psi(t) - a\right) C_f L_2\right) x = 0 \qquad (12)$$

which can be denoted as $\ddot{x} + \alpha_2(t)\dot{x} + \alpha_1(t)x = 0$. Then, by transformation

$$x(t) = \exp\left\{-\frac{1}{2}\int_0^t \alpha_2(t)dt\right\} y(t)$$

Eq. (12) is transformed to be

$$\ddot{y}(t) = -\left\{\alpha_1(t) - \frac{1}{2}\dot{\alpha}_2(t) - \frac{1}{4}\alpha_2^2(t)\right\} y(t)$$

which is a Hill equation. □

Lemma 2 makes it possible to reduce the stability problem of higher order LPTV systems (introduced by the dynamics of SPM gauge) into a second order LPTV system. With the similar proof, Lemma 2 can be generalized for Semi-Proper SISO LPTV SPM Gauge and Slowly Varying SISO LPTV SPM Gauge, due to the application conditions of the iterative solutions of Chang transformation (9)[23][24].

Lemma 3. *With the Semi-Proper SISO LPTV SPM Gauge or the Slowly Varying SISO LPTV SPM Gauge and $T = \pi$, the decoupled slow system of the singularly perturbed Hill equation (6) is still a Hill equation with the same period to the nominal system.*

4.3. SPM Assessment of Hill Equations with SISO LTI SPM Gauge

By Lemma 2, it is known that with an SISO LTI SPM Gauge, the singularly perturbed Hill equation can be decoupled to a fast system and a slow system. The following Theorem 1 is a direct result of the definition for SPM, Lemma 1 and Lemma 2.

Theorem 1. *With SISO LTI SPM Gauge, the SPM of the LPTV nominal system is equal to the maximum singular perturbation parameter of the slow system satisfying*

$$\varepsilon_{\max} = \sup\{\bar{\varepsilon} : \varepsilon \in (0, \bar{\varepsilon})\}$$

when $\varepsilon \in (0, \bar{\varepsilon})$ no eigenvalues of the DTM of the slow system have magnitudes greater than one.

When the perturbation is added in Site 1, design the LTI SPM gauge as a second order minimum-phase fast system, which has a transfer function

$$L_{fast}(s) = \frac{\omega_0^2}{(\varepsilon s)^2 + 2\zeta\omega_0(\varepsilon s) + \omega_0^2}$$

Consider a Mathieu equation, which is a special case of Hill equation.

$$\ddot{y} + \left(a - 2q\cos(2t)\right) y = 0$$

Numerically calculate the DTM for Mathieu equation and check the eigenvalues of the DTM for the stability situation (Lemma 1). The stability diagram is shown by Fig. 2.

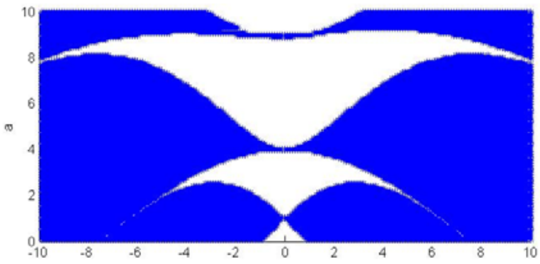

Fig. 2. Stability diagram for the Mathieu equation. Blank regions correspond to stable solutions and shaded regions to unstable solutions. The diagram is symmetrical about *a* axis.

In the expression of the slow system, matrix *L(t)* can be obtained by the iterative expression and the slow system is given by

$$A_{slow} = A_{11} - A_{12}(t)L(t)$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \left(2q\cos(2t) - a\right)\omega_0^2 & 0 \end{pmatrix} L(t)$$
$$L(t) = \sum_{j=0}^{\infty} \varepsilon^j \bar{L}_j(t)$$
$$\bar{L}_0 = A_{22}^{-1}A_{21},\ \bar{L}_{k+1} = A_{22}^{-1}\left[\dot{\bar{L}}_k + \bar{L}_k\left(A_{11} - A_{12}\bar{L}_k\right)\right]$$
$$k = 0, 1, 2, \ldots$$

The parameters are: $a = 3, q = 1, \omega_0 = 2, \zeta = 0.2$.

Numerically calculate the DTM for the slow system and check the eigenvalues of the DTM for the stability situation (Lemma 1). The stability situation is different with different singular perturbation parameter ε, and the results are shown by Table. 2.

Table 2. Stability of the slow system with different singular perturbation parameters (Site 1).

No.	ε	**Max magnitude of the eigenvalues of the DTM**	**Stability**
1.	0.01	1.0	yes
2.	0.03	1.0	yes
3.	0.04	1.0	yes
4.	0.05	1.05	no
5.	0.06	1.06	no

According to Table. 2, the definition of SPM and Theorem 1, it is obtained that $\varepsilon_{\max} = 0.04$.

4.4. SPM Assessment of Hill Equations with SISO LTPV SPM Gauge

The following Theorem 2 and Theorem 3 establish the SPM assessment methods when the Slowly Varying SISO LPTV SPM gauge and the Semi-Proper SISO LPTV SPM gauge are designed, respectively.

Theorem 2 (SPM Assessment of Hill Equation with Semi-proper SISO LPTV SPM Gauge: *Consider the Semi-Proper SISO LPTV SPM Gauge (3) with system period $T=\pi$, and the Parallel Differential - Eigenvalues of coefficient matrix $A_f(t)$, $\mathrm{Re}(\rho_k(t))<-c<0,\ k=1,2,...,m$. The SPM $\varepsilon_{\max}$ of Hill equation (5) can be estimated by $\hat{\varepsilon}_{\max,p}$, which satisfies*

$$\hat{\varepsilon}_{\max,p}=\min\{\varepsilon_1^*,\varepsilon_2^*\}$$

$$\varepsilon_1^*=\sup\left\{\varepsilon^*\ \middle|\ \left|\frac{r}{2}\pm\sqrt{\left(\frac{r}{2}\right)^2-1}\right|<1,\ \forall\varepsilon\in\left[0,\varepsilon^*\right)\right\} \quad (13)$$

where $r=\mathrm{trace}\left\{\phi_{slow,p}(\pi,0,\varepsilon)\right\}$, and $\phi_{slow,p}(\pi,0,\varepsilon)$ is the DTM of pth-order approximated slow system (10), and for the parameter ε_2^*, the function matrix $L_p(t,\varepsilon)$ are the bound and pth-order iterative solution of the DRE for Chang transformation, with an estimate error

$$\left|\hat{\varepsilon}_{\max,p}-\varepsilon_{\max}\right|\to 0,\ \text{when } p\to\infty,\ \text{for } \varepsilon_2^*>\varepsilon_1^*$$

The proof of Theorem 2 relies on the following Lemma 4, which is an important result in its own rights and applicable to not only the second order LPTV systems, but also the general order LPTV systems.

Lemma 4 (Continuity of Floquet Characteristic Multipliers with Respect to Singular Perturbation Parameter): *Consider the Semi-Proper SISO LPTV SPM Gauge (3) with system period $T=\pi$, and the Parallel Differential - Eigenvalues of coefficient matrix $A_f(t)$, $\mathrm{Re}(\rho_k(t))<-c<0,\ k=1,2,...,m$. The Floquet characteristic multipliers of the pth-order approximated slow system $A_{11}(t)-A_{12}(t)L_p(t,\varepsilon)$, $p=1,2,...$ are continuous with respect to the singular perturbation parameter ε.*

Proof of Lemma 4:

When the semi-proper SISO LPTV SPM Gauge (3) is designed in the loop of Fig. 1, from the expressions of singularly perturbed system on Site 1 and Site 2, it can be seen that the system matrix satisfies $A_f(t)=A_{22}(t)$ in the form of the standard singularly perturbed system (6). According to the assumptions that the function matrix $A_f(t)$ satisfies the semi-proper condition (4) and the Parallel Differential-Eigenvalue condition

$$\mathrm{Re}(\rho_k(t))<-c<0,\ k=1,2,...,m$$

Theorem 1 in [23] guarantees that for sufficiently small singular perturbation parameter $\varepsilon<\varepsilon_2^*\le 1$, the iterative form of the DRE in Chang transformation can be written as (9) and the pth-order approximated solutions converge to the exact solutions when $p\to\infty$.

By the continuity of $L(t,\varepsilon)$ and $L_p(t,\varepsilon)$ with respect of ε, function matrices $A_{slow,\infty}(t,\varepsilon)$ *and* $A_{slow,p}(t,\varepsilon)$ are continuous of parameter ε. Furthermore, the DTM of pth-order approximated slow system and the exactly decoupled slow system, denoted by $\phi_{slow,p}(\pi,0,\varepsilon)$ and $\phi_{slow,\infty}(\pi,0,\varepsilon)$, at ε can be written as

$$\begin{aligned}\phi_{slow,p}(\pi,0,\varepsilon)&=W_{slow,p}(\pi,\varepsilon)W_{slow,p}^{-1}(0,\varepsilon)\\ \phi_{slow,\infty}(\pi,0,\varepsilon)&=W_{slow,\infty}(\pi,\varepsilon)W_{slow,\infty}^{-1}(0,\varepsilon)\end{aligned} \quad (14)$$

where $W_{slow,p}(t,\varepsilon)$ and $W_{slow,\infty}(t,\varepsilon)$ are Wronskian matrices of the pth-order approximated slow system and the exactly decoupled slow system, generated from the fundamental solutions of the approximated and exactly decoupled slow systems. Since the solutions of the ordinary differential equation are continuous with respect to the coefficient, Wronskian matrices in (14) and the DTM $\phi_{slow,p}(\pi,0,\varepsilon)$ and $\phi_{slow,\infty}(\pi,0,\varepsilon)$ are continuous with respect to the singular perturbation parameter ε. By the definition of Floquet characteristic multipliers of the LPTV system and since the characteristic polynomial of a matrix is a continuous function of the matrix, the Floquet characteristic multipliers of the pth-order approximated slow system and the exactly decoupled slow system are continuous with respect to ε. □

Proof of Theorem 2:

In [23], it is proved that for sufficiently small singular perturbation parameter, the iterative solutions,

$$\left\|L(t)-L_p(t)\right\|=O(\varepsilon^{p+1}),\quad \left\|H(t)-H_p(t)\right\|=O(\varepsilon^{p+1})$$

and then

$$\left\|A_{slow,p}(t,\varepsilon)-A_{slow,\infty}(t,\varepsilon)\right\|=O(\varepsilon^{p+1})$$

Hence, the pth-order approximated slow system converges to the exactly decoupled slow system $A_{slow,\infty}(t,\varepsilon)$ when $p\to\infty$.

Then, since the solutions of the ordinary differential equation are continuous with respect to the coefficient, by the definitions of Wronskian matrices and the DTM of pth-order approximated slow system and the exactly decoupled slow system (14), we have

$$\left\|\phi_{slow,p}(\pi,0,\varepsilon)-\phi_{slow,\infty}(\pi,0,\varepsilon)\right\|\to 0,\ p\to\infty$$

Since the characteristic polynomial of a matrix is a continuous function of the matrix, we have

$$\left\|\lambda\left(\phi_{slow,p}(\pi,0,\varepsilon)\right)-\lambda\left(\phi_{slow,\infty}(\pi,0,\varepsilon)\right)\right\|\to 0,\ p\to\infty$$

Define the functions $M_p(\varepsilon)$ and $M_\infty(\varepsilon)$ as

$$M_p(\varepsilon)=\max\left|\lambda\left(\phi_{slow,p}(\pi,0,\varepsilon)\right)\right|-1,\quad p=1,2,...$$
$$M_\infty(\varepsilon)=\max\left|\lambda\left(\phi_{slow,\infty}(\pi,0,\varepsilon)\right)\right|-1$$

which satisfy

$$\left|M_p(\varepsilon)-M_\infty(\varepsilon)\right|\to 0,\ \text{ when } p\to\infty \tag{15}$$

By the definition of SPM [20] [21] and Lemma 1, singularly perturbed LPTV system becomes unstable when ε increases to a certain value $\varepsilon_{\max}$, namely,

$$M_\infty(\varepsilon)<0,\quad \text{for } 0<\varepsilon<\varepsilon_{\max}$$
$$M_\infty(\varepsilon)\ge 0,\quad \text{for } \varepsilon\ge\varepsilon_{\max}$$

By Lemma 4, function $M_\infty(\varepsilon)$ is continuous with respect to ε, and there exist a small constant $c>0$ such that

$$M_\infty(\varepsilon)<c,\ \text{ for } 0<\varepsilon<\varepsilon_{\max}$$

Employing Eq. (15), we have

$$M_p(\varepsilon)<c,\quad \text{for } 0<\varepsilon<\varepsilon_{\max},\ p\to\infty$$

By Lemma 1, for $0<\varepsilon<\varepsilon_{\max}$, $p\to\infty$, the Floquet characteristic multipliers of $A_{slow,p}(t,\varepsilon)$ satisfies

$$\left|\lambda_{1,2}\right|=\left|\frac{r}{2}\pm\sqrt{\left(\frac{r}{2}\right)^2-1}\right|<1$$

and the null equilibrium point (origin) of the p^{th} order approximated slow system is exponentially stable. Then, the estimated SPM $\hat{\varepsilon}_{\max,p}$, which is obtained from (13) satisfies

$$\hat{\varepsilon}_{\max,p}\ge\varepsilon_{\max},\ \text{ when } p\to\infty$$

On the other hand, according to Lemma 4, function $M_p(\varepsilon)$ is continuous with respect to ε, it can be proved

$$\hat{\varepsilon}_{\max,p}\le\varepsilon_{\max},\ \text{ when } p\to\infty$$

For $\varepsilon_2^*\ge\varepsilon_1^*$, we have $\hat{\varepsilon}_{\max,p}=\varepsilon_1^*$. Then, the estimate error of the pth-order estimate $\hat{\varepsilon}_{\max,p}$ is

$$\hat{\varepsilon}_{\max,p}\to\varepsilon_{\max},\ \text{ for } \varepsilon_2^*\ge\varepsilon_1^*,\ p\to\infty \qquad \square$$

Theorem 3 (SPM Assessment with Slowly Varying LPTV SPM Gauge): *Consider the Slowly Varying SISO LPTV SPM Gauge (3) with system period $T=\pi$. The SPM $\varepsilon_{\max}$ of Hill equation (5) can be estimated by $\hat{\varepsilon}_{\max,p}$, which satisfies (13), with an estimate error*

$$\left|\hat{\varepsilon}_{\max,p}-\varepsilon_{\max}\right|\to 0,\ \textit{when } p\to\infty,\ \textit{for } \varepsilon_2^*>\varepsilon_1^*$$

The proof of Theorem 3 is similar to the one of Theorem 2, and the slowly varying condition guarantees the convergence of iterative solutions. When the SISO LTI SPM Gauge is given, the following Corollary 1 is a direct result of Theorem 2 and Theorem 3.

Corollary 1 (SPM Assessment with LTI SPM Gauge): Consider the SISO LTI SPM Gauge (3). *The SPM $\varepsilon_{\max}$ of Hill equation (5) can be estimated by $\hat{\varepsilon}_{\max,p}$, which satisfies (13), with an estimate error*

$$\left|\hat{\varepsilon}_{\max,p}-\varepsilon_{\max}\right|\to 0,\ \textit{when } p\to\infty$$

4.4. SPM Assessment of General Order LPTV Equations

The DTM, denoted by $\phi(\theta,0)$, is a key point for the stability margin assessment, and according to Floquet theory

$$x(t+m\theta)=\phi(\theta,0)^m x(t)$$

let $t=0$ and $m=0,1,2,...,n$ in the above equation, through simulation to know the value of $x(0),x(\theta),...,x(n\theta)$, and then the elements of $\phi(\theta,0)$ becomes an n^2 algebraic equation.

Due to the numerical methods for DTM, the algorithms of Section 4.3 and Section 4.4 can be generalized to any order LPTV system to obtain the singular perturbation margin.

4.5. GGM Assessment of Hill Equations

The GGM is essentially a measure of regular perturbations. Due to the explicit DTM expressions of second order periodic systems, Lemma 1 can be directly used for the GGM assessment of perturbed system $\dot{x}=A(t,k)x$ by the definition of GGM. Consider Meissner Equation and Hill equation with impulse coefficient as examples, as shown by Fig. 3.

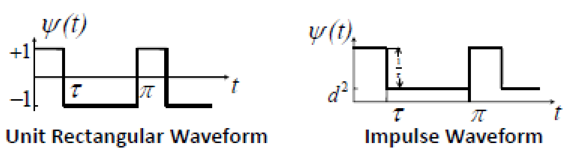

Fig. 3. Meissner Equation and Impulse Coefficient Hill Equation.

GGM Assessment of Meissner Equation.

Meissner Equation is a well-known Hill equation, when the function $\psi(t)$ is unit rectangular waveform in (5), which can be viewed as the pair of constant coefficient equations.

$$\ddot{x}+(a-2q)x=0, \quad 0\le t<\tau$$
$$\ddot{x}+(a+2q)x=0, \quad \tau\le t<\pi$$

The DTM of Meissner equation is given by

$$\phi(\pi,0)=\begin{bmatrix}\cos d(\pi-\tau) & \frac{1}{d}\sin d(\pi-\tau)\\ -d\sin d(\pi-\tau) & \cos d(\pi-\tau)\end{bmatrix}\begin{bmatrix}\cos c\tau & \frac{1}{c}\sin c\tau\\ -c\sin c\tau & \cos c\tau\end{bmatrix}$$

where $d=\sqrt{(a+2q)}, c=\sqrt{(a-2q)}$.

Case 1. With constant regular perturbation parameter *k* in the closed loop, either Site 1 or Site 2 in Fig. 1, the DTM of the regularly perturbed system is given by the above $\phi(\pi,0)$, albeit the parameters *k* dependent. Denote

$$\bar{d}=d\sqrt{k}=\sqrt{(a+2q)k},\ \bar{c}=c\sqrt{k}=\sqrt{(a-2q)k}$$

The characteristic multiplier can be written as

$$\lambda_{1,2}=r/2\pm\sqrt{(r/2)^2-1}$$

and

$$r=\mathrm{trace}\{\phi_{pert,k}(\pi,0)\}=2\cos\bar{d}(\pi-\tau)\cos\bar{c}\tau-\left(\frac{\bar{d}}{\bar{c}}+\frac{\bar{c}}{\bar{d}}\right)\sin\bar{d}(\pi-\tau)\sin\bar{c}\tau$$

By Lemma 1, the stability diagram for Meissner equation with GGM parameter *k* is shown by Fig. 4.

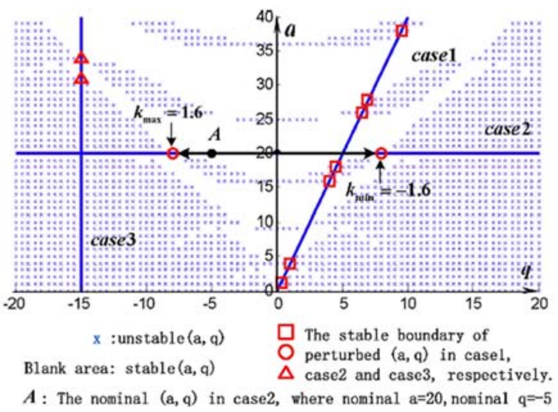

Fig. 4. Stability Diagram for Meissner Equation and GGM Assessment.

In Fig. 4, the blank area represents the stable solutions of Meissner equation with the parameter couple (q,a) . When the nominal system with parameter couple (q_0,a_0) , is perturbed by regular perturbation parameter *k*, then (q,a) will move along the line $a/q=3$, until the couple (q_f,a_f) first touches the boundary of the blank area, i. e. the regularly perturbed system becomes marginally stable, which is shown by the legend square. Then, the GGM $k_{\max}$ and $k_{\min}$ are obtained.

Case 2. When Meissner equation is regularly perturbed on parameter *q*. The parameters $\bar{d}$ and $\bar{c}$ in the DTM of the regularly perturbed system become $\bar{d}=\sqrt{a+2qk}$, $\bar{c}=\sqrt{a-2qk}$. In Fig. 4, when $a=20$, the legend circles show boundaries of stability solutions, and with $q=-1$ as the nominal system parameter (point *A*), the GGM of Meissner equation is $k_{\max}=1.6$, $k_{\min}=-1.6$.

Case 3. When Meissner equation is regularly perturbed on parameter *a*, the parameters $\bar{d}$ and $\bar{c}$ in the DTM become $\bar{d}=\sqrt{ak+2q}$, $\bar{c}=\sqrt{ak-2q}$. In Fig. 3, the legend triangle represent the boundaries of stable solutions.

GGM Assessment of Hill Equation with an Impulsive Coefficient

When the function $\psi(t)$ is impulsive waveform in (5), the DTM is given by

$$\phi(\pi,0)=\begin{bmatrix}\cos d\pi-\frac{1}{d}\sin d\pi & \frac{1}{d}\sin d\pi\\ -(d\sin d\pi+\cos d\pi) & \cos d\pi\end{bmatrix},$$

where $d=\sqrt{(a+2q)}$.

Case 1. With constant regular perturbation parameter *k* in the closed loop, the DTM of the regularly perturbed system is given by the above $\phi(\pi,0)$, albeit the parameters *k* dependent. Denote

$$\bar{d}=d\sqrt{k}=\sqrt{(a+2q)k},$$

The characteristic multiplier can be written as

$$\lambda_{1,2}=r/2\pm\sqrt{(r/2)^2-1}$$

where

$$r=\mathrm{trace}\{\phi_{pert,k}(\pi,0)\}=2\cos\bar{d}\pi-\frac{1}{\bar{d}}\sin\bar{d}\pi$$

By Lemma 1, the stability diagram for Hill Equation with an impulsive coefficient and the regular perturbation parameter *k* is shown from Fig. 5. The legend square shows the GGM when $a/q=2.5$.

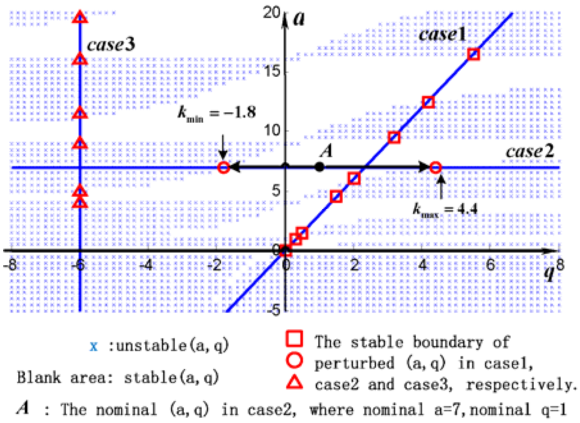

Fig. 5. Stability Diagram of Hill Equation with an Impulsive Coefficient.

Case 2. When parameter q is perturbed on, the parameter $\bar{d}$ in the DTM of the regularly perturbed system become $\bar{d} = \sqrt{a+2qk}$. Fig. 5 shows when $a = 7$ (the legend circle shows boundaries of stability), with $q = 1$ as the nominal system parameter (Point A), the GGM of Hill equation is

$$k_{\max} = 4.4,\ k_{\min} = -1.8$$

Case 3. When parameter a is regularly perturbed, the parameter $\bar{d}$ in the DTM becomes $\bar{d} = \sqrt{ak+2q}$. In Fig. 5, the legend triangle represent the boundaries of stable solutions.

4.6. GGM Assessment of General Order Equations

Due to the numerical methods for DTM, the algorithms of section 4.5 can be generalized to any order LPTV system to obtain the GGM of the nominal system (1).

5. Example: Inverted Flexible Beam

This example is to illustrate the correctness of the SPM and GGM assessment methods. The Inverted Flexible Beam model, shown by Fig. 6, is deduced from the original horizontal Flexible Beam model in [25] by adding a gravity term, and denote motor angle θ, angular tip deflection $\alpha = d/l$, tip deflection d, and link length l.

Assume that the interaction moment between hub and beam can be represented as $M_{\text{in}} = C_{\text{stiff}}\alpha$ .The differential equations for this setup are

$$\begin{aligned} \ddot{\theta} &= \frac{C_{\text{stiff}}}{J_m}\alpha + \frac{T}{J_m} \\ \ddot{\alpha} &= -\frac{C_{\text{stiff}}(J_l+J_m)}{J_m J_l}\alpha + \frac{lmg}{2J_l}\sin\theta - \frac{T}{J_m}, \end{aligned} \tag{16}$$

where T is the torque applied by the motor which is proportional to the motor voltage. The physical parameters are given by Table 3 [25].

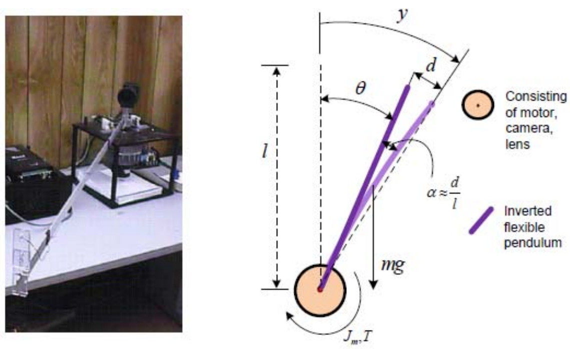

Fig. 6. A Flexible Inverted Pendulum.

Table 3. Physical Parameters.

Physical Parameters	Symbol	Numerical Value
Beam Length	l	1.21 m
Beam Mass	m	0.2639 kg
Bending Stiffness	E	$6.9\times10^{10}\,N/m^2$
Beam Section Height	h	$3.4\times10^{-3}\,m$
Beam Section Width	b	$2.54\times10^{-2}\,m$
Beam Stiffness	C_{stiff}	$8.5736\ kgm^2/s^2$
Hub Inertia	J_m	$6.4\times10^{-3}\,kgm^2$
Beam Inertia	J_l	$9.66\times10^{-2}\,kgm^2$

When the trajectory tracking problem is formulated, let $\bar{\alpha}, \bar{\theta}, \bar{T}$ be the nominal state, output and input satisfying Eq. (16). Define the state, output and input of the error dynamics as

$$\tilde{\alpha} = \alpha - \bar{\alpha},\ \tilde{\theta} = \theta - \bar{\theta},\ \tilde{T} = T - \bar{T}$$

and then the linearized tracking error dynamics can be written as

$$\begin{bmatrix} \dot{\tilde{\theta}} \\ \dot{\tilde{\alpha}} \\ \ddot{\tilde{\theta}} \\ \ddot{\tilde{\alpha}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{C_{\text{stiff}}}{J_m} & 0 & 0 \\ \dfrac{lmg}{2J_l} & -\dfrac{C_{\text{stiff}}(J_l+J_m)}{J_m J_l} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\alpha} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{\alpha}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{J_m} \\ -\dfrac{1}{J_m} \end{bmatrix} \tilde{T}$$

SPM and GGM Calculation: The eigenvalues of the closed loop system are designed as

$$\begin{aligned} \bar{\rho}_{1/2} &= (-0.4240 \pm 1.2630\,j)a \\ \bar{\rho}_{3/4} &= (-0.6260 \pm 0.4141\,j)a,\ a = 13 \end{aligned}$$

and the system is stabilized. Using a 2nd-order LPTV SPM Gauge in the α loop expressed by

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -\frac{(\omega_0 \sin(\gamma t))^2}{\varepsilon} z_1 - \frac{2\zeta_0(\omega_0 \sin(\gamma t))^2}{\varepsilon} z_2 + u$$
$$y = (\omega_0 \sin(\gamma t))^2 z_1$$

When $\gamma = 0.5\ rad/s$, $\omega_0 = 1$, $\zeta_0 = 1$, the system matrix of the singularly perturbed system can be written as

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{J_m} & \frac{C_{\text{stiff}}}{J_m} & \frac{k_3}{J_m} & \frac{k_4}{J_m} & \frac{k_2(\omega_0 \sin(\gamma t))^2}{J_m} & 0 \\ \frac{lmg}{2J_l} - \frac{k_1}{J_m} & -\frac{C_{\text{stiff}}(J_l + J_m)}{J_m J_l} & -\frac{k_3}{J_m} & -\frac{k_4}{J_m} & -\frac{k_2(\omega_0 \sin(\gamma t))^2}{J_m} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 1 & 0 & 0 & -\frac{\omega_0^2 \sin(\gamma t)}{\varepsilon} & -\frac{2\zeta_0 \omega_0 \sin(\gamma t)}{\varepsilon} \end{bmatrix},$$

where $k_i, i = 1,2,3,4$ are the designed controller parameters. Through numerical calculation of the DTM and the Floquet characteristic multipliers, when $\varepsilon \in (0, 0.0113]$, the maximum characteristic multiplier modulus is $|\lambda|_{\max} = 0.9949 \le 1$. According to Lemma 1, the LPTV system is stable. If $\varepsilon > 0.0113$, it is obtained $|\lambda|_{\max} > 1$. Thus, the SPM of this closed loop system is $\varepsilon_{\max} = 0.0113$. Fig. 7 shows the simulation results when $\varepsilon = 0.011, 0.015$.

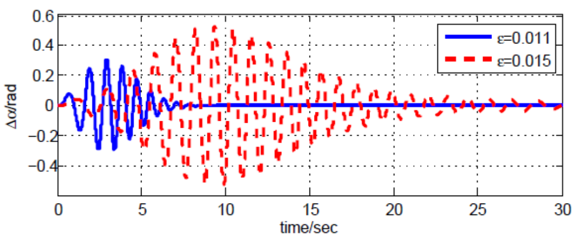

Fig. 7. The Simulation Result of $\tilde{\alpha}$.

With a time-varying periodical GGM Gauge $k\sin(\gamma t)$ in the $\tilde{\alpha}$ feedback loop, the regularly perturbed system matrix can be derived as

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{J_m} & \frac{C_{\text{stiff}} + k_2 k}{J_m} & \frac{k_3}{J_m} & \frac{k_4}{J_m} \\ \frac{lmg}{2J_l} - \frac{k_1}{J_m} & -\frac{C_{\text{stiff}}(J_l + J_m)}{J_m J_l} - \frac{k_2 k}{J_m} & -\frac{k_3}{J_m} & -\frac{k_4}{J_m} \end{bmatrix}$$

When $k \in [-1.4003, 1.4003]$, the maximum characteristic multiplier modulus is

$$|\lambda|_{\max} = 0.999 \le 1$$

Then the GGM is

$$k_{\max} = 1.4003, k_{\min} = -1.4003$$

The calculation of the GGM and the simulation results are shown by Fig. 8.

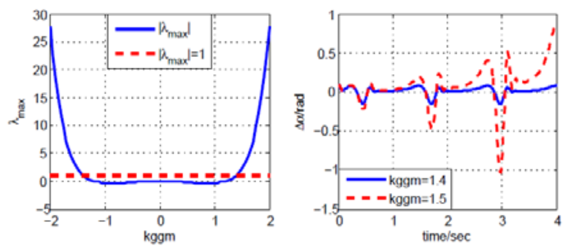

Fig. 8. Left: $|\lambda|_{\max}$ with different *k*. Right: The Simulation Result of $\tilde{\alpha}$ with different *k*.

6. Conclusions

The problem of SPM and GGM assessment for LPTV systems is formulated, which provides a necessary step for the development of the theoretically based and practically efficient SPM and GGM analysis methodology for Nonlinear Time-Varying (NLTV) systems in the near future.

Chang transformation makes it possible to reduce the SPM analysis for Hill equations, which is essentially a stability problem of higher order LPTV systems due to the SPM gauge introduced dynamics, to second order LPTV systems.

Based upon Floquet Theory, when different types of LPTV SPM gauges are designed, the SPM and GGM assessment methods for the second order and general order LPTV systems are established and stated by the corresponding lemmas, theorems and corollaries. The effectiveness of such methods are illustrated by an examples of a flexible inverted pendulum.

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