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Research on an Approach to Ultrasound Diffraction Tomography Based on Wave Theory

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Abstract: According to the ultrasonic propagation characteristics in continuous mediums, the propagation problem of sound waves in inhomogeneous mediums was studied. Based on the scattering field wave equation and total field wave equation, the method of moments was introduced into the discrete problem of equations which provided a theoretical basis for solving wave equations. The Born iterative algorithm was adopted to solve the nonlinear problem of equations. The truncated singular value decomposition regularization method was adopted to solve the stability of solutions. Experimental results indicate that the above methods can be applied to the strong scattering field and have higher image quality. *Copyright © 2013 IFSA.*

Keywords: Wave equation, Iterative algorithm, Truncated singular value decomposition, Ultrasound computed tomography.

1. Introduction

In the past twenty years, ultrasonic diffraction tomography technology develops rapidly. The core of the technology is based on iterative to approach the truth. The equation based on wave theories usually has an ill-posed problem which is about existence, uniqueness and stability of the solution. For the uniqueness and existence, we can use appropriate mathematics method to solve. For the stability of the solution, we should use a regularization method to solve it [1, 2].

2. Establishment of Wave Equations

In tomographic imaging, the object is a homogeneous medium. We are interested in the sound waves through the heterogeneity of medium, so the wave equations in inhomogeneous mediums must be studied [3].

Medium internal refraction coefficient change function $k(\vec{r})$ can be indicated:

$$k(\vec{r}) = \frac{\omega}{c(\vec{r})} = \frac{\omega}{c_0} \cdot \frac{c_0}{c(\vec{r})} = k_0 n(\vec{r}) \qquad (1)$$

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where $n(\vec{r}) = \dfrac{c_0}{c(\vec{r})}$ is the refraction coefficient of the point $\vec{r}$, k_0 is the average number of waves in the medium, $k_0 = \dfrac{\omega}{c_0}$, c_0 is the propagation velocity of the sound wave in an uniform medium, $c(\vec{r})$ is the velocity of the sound wave about the internal point $\vec{r}$, ω is acoustic angular frequency.

Wave equation can be transformed into the following form:

$$(\nabla^2 + k_0^2) p(\vec{r}) = -k_0^2 (n^2(\vec{r}) - 1)) p(\vec{r}) \quad (2)$$

∇^2 is LABOLAS operator, $p(\vec{r})$ is the pressure when the sound wave propagate in the medium. If $o(\vec{r}) = k_0^2 \left[n^2(\vec{r}) - 1 \right]$, then Eq. (2) can be written as

$$(\nabla^2 + k_0^2) p(\vec{r}) = -o(\vec{r}) p(\vec{r}) \quad (3)$$

Eq. (3) is the inhomogeneous wave equation when ultrasonic wave propagates in the non-uniform medium. Ultrasonic propagation in the non-uniform medium generates the filed $p(\vec{r})$. The filed can be divided into two parts: the scattering filed $p_s(\vec{r})$ and the incident field $p_0(\vec{r})$.

$$p(\vec{r}) = p_0(\vec{r}) + p_s(\vec{r}) \quad (4)$$

$p_0(\vec{r})$ is the acoustic field when the acoustic wave is between emitter and receiver propagates in the uniform medium. It fits to the equation $[\nabla^2 + k_0^2] p_0(\vec{r}) = 0$. While $p_s(\vec{r})$ is in the case of non-uniform medium and it fits to $(\nabla^2 + k_0^2) p_s(\vec{r}) = -o(\vec{r}) p_s(\vec{r})$.

From Eq. (4), we can get the scattering field by Green's Function $G(\vec{r} - \vec{r}')$. Supposing the scattering field as scattering point array, the scattering field can be showed as

$$p_s(\vec{r}) = \int G(\vec{r} - \vec{r}') o(\vec{r}') p(\vec{r}') d(\vec{r}') \quad (5)$$

Eq. (5) means that the scattering field can be showed by the superposition of scattering point source.

So the whole field equation can be written as

$$p(\vec{r}) = p_0(\vec{r}) + p_s(\vec{r}) = p_0(\vec{r}) + \int G(\vec{r} - \vec{r}') o(\vec{r}') p(\vec{r}') d(\vec{r}') \quad (6)$$

Eq. (5) and Eq. (6) describe characteristics of the scattering field and the whole field .They are the base of ultrasound CT.

Using $\varphi_j = \delta(\vec{r} - \vec{r}')$ to make $o(\vec{r}) p(\vec{r})$ become

$$o(\vec{r}) p(\vec{r}) = \sum_j o(\vec{r}_j) p(\vec{r}_j) \varphi_j \quad (7)$$

$j = 1,2,3,\cdots,N$, N is the amount of divided units.

Substituting Eq. (7) into Eq. (6), we can get

$$p(\vec{r}) = p_0(\vec{r}) + \sum_j o(\vec{r}_j') p(\vec{r}_j') \int G(\vec{r} - \vec{r}') dr \quad (8)$$

Choosing Dirac function δ as the test function and making inner product with each end of Eq. (8), then the whole field of any point in the rectangular area can be

$$p(\vec{r}_i) = p_0(\vec{r}_i) + \sum_j o(\vec{r}_j') p(\vec{r}_j') \int G(\vec{r}_i - \vec{r}') \delta(\vec{r}' - \vec{r}_j) d\vec{r}' \quad (9)$$

In Eq. (9), $i = 1,2,3,\cdots,N$.

By the characteristics of δ function, Eq. (9) can be transformed into

$$p(\vec{r}_i) = p_0(\vec{r}_i) + \sum_j o(\vec{r}_j') p(\vec{r}_j') \int_{\Omega_j} G(\vec{r}_i - \vec{r}') d\vec{r}', \quad (10)$$

where Ω_j is the j th rectangular unit which is divided into.

The two-dimension green function can be showed by the first kind of HANKEL function. By the characteristics of HANKEL function, we can get

$$\int_{\Omega_j} G(\vec{r}_i - \vec{r}') d\vec{r}' = \begin{cases} \dfrac{j}{2}[\pi k_0 a H_1^{(2)}(k_0 a - 2j] & i = j, \\ \dfrac{j\pi k_0 a}{2} J_1(k_0 a) H_0^{(2)}(k_0 R_{ij}] & i \neq j \end{cases} \quad (11)$$

where J_1 is the first kind of Bessel function, $H_n^{(m)}$ is the m[th] class n order BESSEL function, a is the small unit inscribed circle radius of the rectangular area, R_{ij} is the distance of the r_i and r_j.

Supposing

$$c_{ij} = \begin{cases} \dfrac{j}{2}[\pi k_0 a H_1^{(2)}(k_0 a - 2j] & i = j \\ \dfrac{j\pi k_0 a}{2} J_1(k_0 a) H_0^{(2)}(k_0 R_{ij}] & i \neq j \end{cases}$$

k_0 is the wave number of the ultrasonic waves in the water, $k_0 = 2 * \pi / L$, a is the inscribed circle radius of the discrete small area, $a = d/2$, d is sampling interval of the square area for uniform sampling, $d = L/10$, L is the wavelength.

Eq. (10) can be simplified as

$$p(\vec{r}_i) = p_0(\vec{r}_i) + \sum_j o(\vec{r}_j^{'}) p(\vec{r}_j^{'}) c_{ij} \qquad (12)$$

Eq. (12) is the total field equation after scattering [4].

Supposing $p^{(t)} = p(\vec{r}_i)$ is $N \times 1$ dimension total field vector, $p^{(in)} = p_0(\vec{r}_i)$ is $N \times 1$ dimension incident field vector, $O = diag(o(\vec{r}_j^{'}))$ is the diagonal matrix of the unknown function, $C = c_{ij}$ is the forward scattering equation's $N \times 1$ dimension coefficient matrix, so Eq. (12) can be simplified as:

$$P^{(t)} = P^{(in)} + COP^{(t)} \qquad (13)$$

The scattering field Eq. (7) can be discrete as the follow form through the same method:

$$p_s(\vec{r}_m) = \sum_j o(\vec{r}_j^{'}) p(\vec{r}_j^{'}) d_{mj} \qquad (14)$$

In Eq. (14), $d_{mj} = \frac{j\pi k_0 a}{2} J_1(k_0 a) H_0^{(2)}(k_0 R_{mj}]$, $m = 1,2,3,\cdots,M$ is the receiving scattering field point. $p^{(s)} = p_s(\vec{r}_m)$ is $M \times 1$ dimension scattering field vector, $D = d_{mj}$ is $M \times N$ dimension scattering equation's coefficient matrix. Eq. (14) can be written as [5].

$$P^{(s)} = DOP^{(t)} \qquad (15)$$

3. Nonlinear Problem of Wave Equations

Ultrasonic tomography belongs to inverse scattering problems. Though Eq. (14) gives the relation between the scattering field and unknown function, but because the field equation is also the unknown function's equation which leads to Eq. (12) and Eq. (14) are the nonlinear equations. Usually the iterative method is adopted to solve the nonlinear to achieve the purpose of approximation object interior acoustic parameter distribution [6]. At present, the method to solve nonlinear problems can be roughly divided into two kinds. One kind is transforming the nonlinear equation into two linear problems and then does iteration between the two linear equations. Born iterative belongs to this kind of method. The other kind is that people can use the nonlinear optimization method to solve the problem, such as L-M method.

The fundamental idea of Born iterative algorithm is that the solution of unknown function be more close to the actual solution gradually based on the scattering field equation and the total field equation. In this algorithm, the coefficient matrix is constant in each iteration process. Along with the iteration, the change of unknown function means the scattering body changing and the GREE function represents the function about scattering characteristics of a point source. Therefore, with the iteration, we can accord to the unknown function's change to correct the green's function. This means that the coefficient matrix of inverse scattering equation can recount in every iteration process. This method makes the solution of unknown function be more close to the actual solution [7].

4. Stability Problem of Wave Equations

The inverse scattering problem which is established on the basis of wave theory can be attributed to the ill-posed equation [8]. The core equation in the actual solving process of Born iterative method is $\Delta P_k^{(s)} = D\Delta O_k P_k^{(t)}$. In order to avoid the multifarious mark in the process, we need to simplify the equation. Firstly, the symbol position of the equation is transformed, it can be written as $\Delta P_k^{(s)} = D[[P_k^{(t)}]]\Delta O_k$, $[[P_k^{(t)}]]$ is the diagonal matrix of $P_k^{(t)}$, ΔO_k is the increment of the unknown function.

Supposing

$$x \cong \Delta O_k$$
$$A \cong D[[P_k^{(t)}]]$$
$$b = \Delta P_k^{(s)}$$

The equation is transformed into a common form:

$$Ax = b \qquad (16)$$

When the data item b has a small perturbation, the smaller singular value has the amplification effect on the disturbance which can account for results in the solution deviating from the real solutions. The method to solve this kind of ill-posed problem is the regularization which is looking for a well-posed equation to approach the problem and make the well-posed equation solution approach the original true solution.

In this paper the truncated singular value decomposition regularization method is applied to solve the inverse scattering problem. It can better the noise of filter and improve the quality of reconstruction images. It also can reduce the iteration calculation. The method of truncated singular value decomposition is directly transforming the morbid matrix A . Its main idea is looking for a well-conditioned matrix A , so it can well approach the matrix A under the 2 - norm.

On the basis of singular value decomposition theory, for any $k < rank(A)$, if we take

$$A_k = \sum_{i=1}^{k} u_i \sigma_i v_i^T \tag{17}$$

then

$$\min_{rank(B)=k} \|A - B\|_2 = \|A - A_k\|_2 \tag{18}$$

A_k is the matrix which is the closest to the original matrix among the rank k matrices. Therefore, we can transform the solution of original problem $Ax = b$ into the well-posed equation $A_k x = b$ .The solution of $A_k x = b$ is

$$x_{tsvd} = \sum_{i=1}^{k} \frac{\langle u_i, b \rangle}{\sigma_i} v_i \tag{19}$$

The above formula is equivalent to directly cut the smaller singular value of least squares solution of the equation. So the method is called the truncated singular value decomposition method (the abbreviation is *TSVD*) [9].

5. Experimental Results and Analysis

The original image is shown in Fig. 1. Its internal structure is complicated.

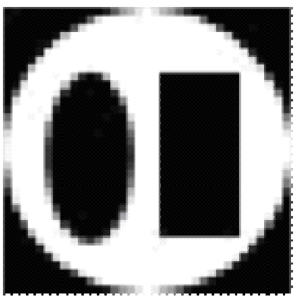
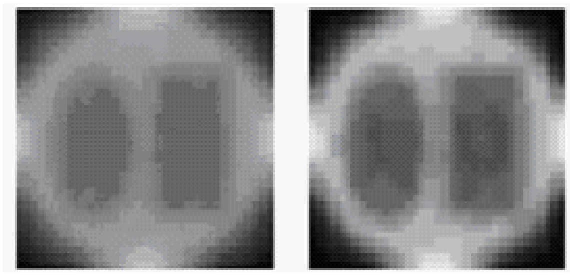


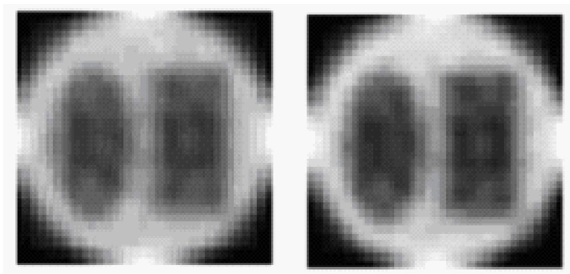
Fig. 1. Original image.

The sensors are evenly arranged on a circle which is around the object. In order to avoid the influence of instant disappeared waves, the circle radius is usually taken for 20λ . The frequency for 200 KHZ ultrasonic is used as the incident wave. We make uniform sampling of the square area, the sampling interval is $\lambda/10 = 0.75\,mm$, the number of sampling points is for 35×35. In order to collect enough scattering information, we evenly arrange fifty sensors in the ring which is around the object. Then we can get inverse scattering equations which have 1225 unknown numbers.

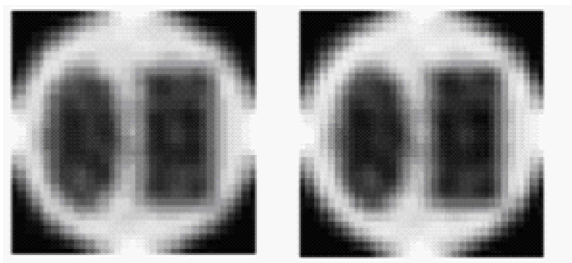
In the case of contrast 20 %, a sensor as the emission sensor launches ultrasonic signal, the rest of sensors are as receiving sensors. Under the control of pulse control circuit, each sensor acts in turn as the emission sensor, the rest of sensors acts as receiving sensors. After the receiving sensors receive ultrasonic signals, the reconstruction program is adopted to complete the computed tomography according to data.



(a) Iterations for 1 (b) Iterations for 2



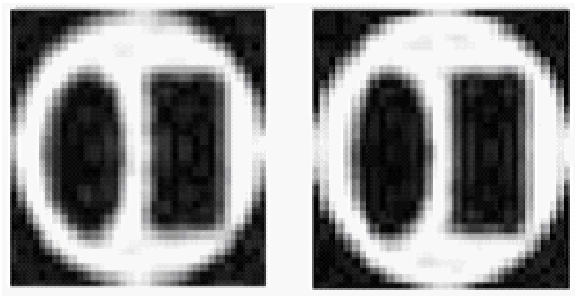
(c) Iterations for 3 (d) Iterations for 4



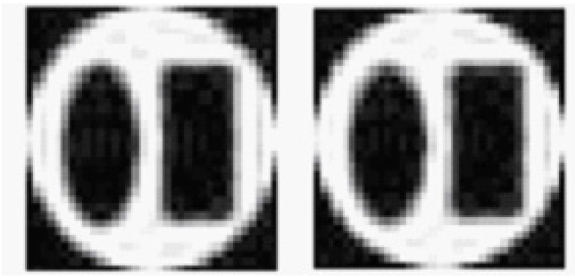
(e) Iterations for 5 (f) Iterations for 6

Fig. 2. Reconstruction results with non-regularization.

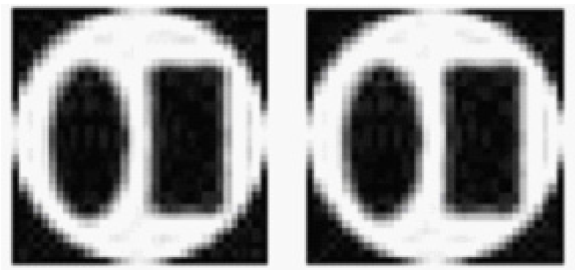
According to the received data, we can reconstruct fifty images about the unknown function. The plenty of information about the unknown function can be acquired through the sensors which are evenly arranged around the object. Though doing average processing to the fifty images, we get rich information of the unknown function images. Fig. 2 is about the images after several iterations. We can see from Fig. 2 that the effect is very poor after the first iterative. Along with the increase of iteration times, the reconstruction effect is becoming better and better.



(a) Iterations for 1 (b) Iterations for 2



(c) Iterations for 3 (d) Iterations for 4



(e) Iterations for 5 (f) Iterations for 6

Fig. 3. Reconstruction results based on TSVD.

Fig. 3 is about the reconstruction results based on TSVD. From the figure we can clearly see the material internal section status and the reconstruction image is much closer to the original image.

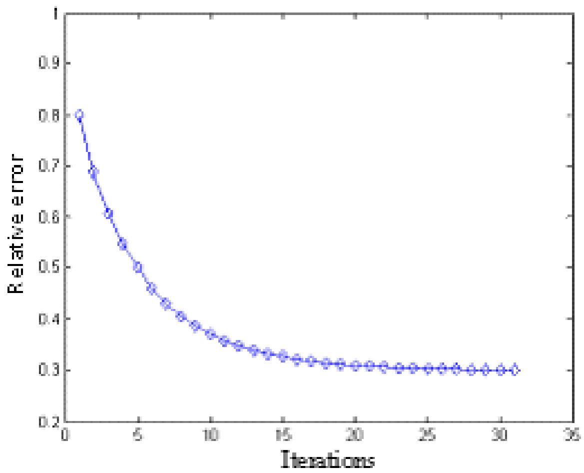
The relative error can be used to evaluate reconstruction images. It indicates the accurate degree of solution:

$$RE = \frac{\left\| x_{reg} - x_{ext} \right\|_2}{\left\| x_{ext} \right\|_2} \tag{20}$$

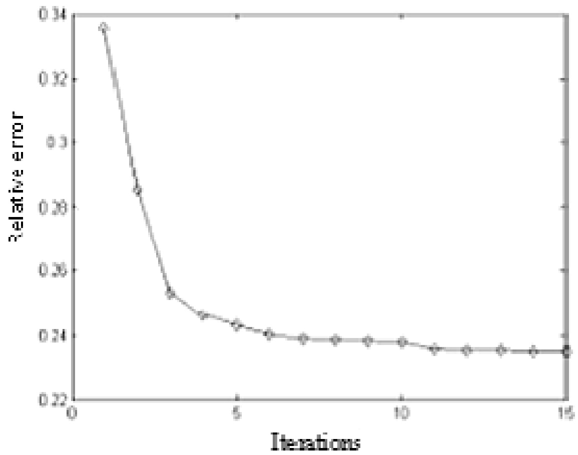
In Eq. (20), x_{reg} stands for the solution of the equation, x_{ext} is the accurate solution about the original image.

Table 1. Relative error.

	Results based on average processing	Results based on TSVD
Iterations for 1	0.8002	0.3355
Iterations for 2	0.6866	0.2835
Iterations for 3	0.6073	0.2526
Iterations for 4	0.5473	0.2455
Iterations for 5	0.5003	0.2425
Iterations for 6	0.4627	0.2396



(a) Curve with non-regularization



(b) Curve based on TSVD

Fig. 4. Relation curve of relative error and iterations.

Through the comparison about relative error, we can see from Fig. 4: with the increase of iteration times, the relative error given by the two methods tends to be stable. The relative error based on TSVD method is far less than the relative error which does not use the regularization method. The experiment shows that the reconstruction results based on the TSVD method are better than the non- regularization method in the same iterations.

6. Conclusions

On the basis of wave theory, we have researched the discrete problem of equations about the whole and scattering field. These are the foundation of scattering CT imaging. The iterative algorithm is adopted to reconstruct images. The experimental model adopts the around type array structure. The ill-posed problem is solved by using the truncated singular value decomposition regularization method. This method can accelerate the convergence speed of equations.

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