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Embedding the Photon with Its Relativistic Mass as a Particle into the Electromagnetic Wave Explains the Gouy Phase Shift as an Energetic Effect

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Abstract: A new aspect concerning the relationship between photon and electromagnetic wave has been developed by considering the question why the energy and the mass density of an electromagnetic wave are propagating in the same direction. For instance, in optical resonators the energy density usually propagates along curved lines. However, according to Newton's first law the mass density should propagate along a straight line, if no force is exerted it. In order to solve this problem, the assumption has been made that a transverse force is exerted on the mass density and in consequence on the mass of the photons which forces them to follow the propagating energy density. The expression obtained for this force makes it possible to show that the photon is moving within a transverse potential. This finally leads to the result that the transverse probability density of the photons can be computed by solving a Schrödinger equation identical with the Schrödinger equation describing the motion of the electron, except that the mass of the electron is replaced by the relativistic mass of the photon. In this way, it could for the first time be shown that the Schrödinger equation is also describing the motion of a particle which has no rest mass. The results obtained in this way for the intensity distribution of Gaussian waves and the Gouy phase shift are in full agreement with the results obtained by the use of wave optics. In addition, the Gouy phase shift could be explained as an energetic effect. For this purpose, it has been shown that the energy, which is obtained by multiplying the effective axial propagation constant $\bar{k}_z(z)$ by ħc, is equal to the difference between the total energy E_{ph} of the photon and the energy eigenvalues $E_{nm}(z)$ of the Schrödinger equation.

Keywords: Quantum optics, Electromagnetic wave, Paraxial wave optics, Physical optics, Laser theory, Laser resonators.

1. Introduction

The relation between photon and electromagnetic wave has been considered from many aspects [1]; however, no explicit mathematical expression has been derived describing how the photon is embedded as a particle into the propagating electromagnetic wave. In contrary, David Bohm has argued in his quantum theory book [2] that there is no quantity for light equivalent to the electron probability density $P_e(x) = |\psi(x)|^2$. In particular, he is claiming [2]: "There is, strictly speaking, no function that represents the probability of finding a light quantum at a given point". In the following, based on a consideration of the question why the energy and the mass density of an electromagnetic wave are propagating in the same direction, nevertheless, the attempt is made to embed the photon with its relativistic mass as a particle into the electromagnetic wave.

In optical resonators the energy density usually propagates along curved lines, as shown in Fig. 1.

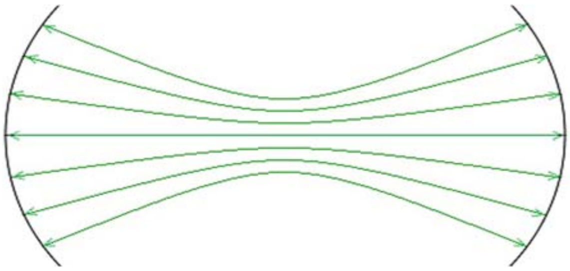

Fig. 1. Resonant Gaussian mode between two spherical mirrors. The green lines visualize the propagating energy density.

However, according to Newton's first law the mass density should propagate along a straight line, if no force is exerted it. In order to solve this problem, the assumption has been made that a transverse force is be exerted on the mass density and in consequence on the mass of the photons which forces them to follow the propagating energy density [3-4]. To compute this force, it has been taken into account that the directional change of the propagating energy density is described by the directional change of the Poynting vector which describes the directional energy flux of the electromagnetic wave. In this way, a mathematical expression for the force is obtained by considering an infinitesimal propagation step of the electromagnetic wave. The expression obtained for this force makes it possible to show that the photon moves within a transverse potential which in combination with a Schrödinger equation allows to describe the transverse quantum mechanical motion of the photon by the use of matter wave theory like the motion of the electron, even though the photon has no rest mass. The obtained results are verified for the plane, the spherical and the Gaussian wave. An additional verification could be provided by the fact that also the mathematical equation describing the Guoy phase shift [5] could be derived from this quantum mechanical particle picture in full agreement with wave optics. Moreover, the particle picture allows to explain the Gouy phase shift as an energetic effect. For this purpose, it has been shown that the energy $\hbar c\bar{k}_z(z)$, which is obtained by multiplying the effective axial propagation constant $\bar{k}_z(z)$ by ħc, is equal to the difference E_{ph}-$E_{nm}(z)$ between the total energy E_{ph} of the photon and the energy eigenvalues $E_{nm}(z)$ of the Schrödinger equation. Since according to this result $\hbar c\bar{k}_z(z)$ represents a real energy, it has been concluded that also the effective axial propagation constant $\bar{k}_z(z)$ represents a real propagation constant. This leads to the conclusion that $\lambda_{nm}(z) = 2\pi/\bar{k}_z(z) = hc/\left(E_{ph} - E_{nm}(z)\right)$ represents the real local wave length of the photon at the position z. According to this conclusion $\lambda_{nm}(z)$ increases inversely proportional to $\bar{k}_z(z)$, and therefore, describes the Gouy phase shift in agreement with wave optics. But the above expression also shows that $\lambda_{nm}(z)$ increases inversely proportional to energy difference $E_{ph} - E_{nm}(z)$ which decreases with decreasing z. This shows that the deeper physical reason for Gouy phase shift seems to consist in the fact that the energy of the photon is increasingly converted into its transverse quantum mechanical motion when the photon approaches the focus. This explains the Gouy phase shift as an energetic effect.

2. Derivation of a Transverse Force Exerted on a Photon Propagating with an Electromagnetic Wave

As well-known from the propagation of laser beams, the energy density, whose propagation follows the Poynting vector, is usually not propagating along a straight line as shown by Fig. 1 which displays a resonant Gaussian wave between two spherical mirrors of an optical resonator. Therefore, the question arises, what is happening with a particle of mass propagating with the wave. Since it simultaneously represents a quantum of energy its propagation direction should change in agreement with the changing direction of the Poynting vector as shown by the green lines in Fig. 1. However, if the quantum of energy is considered to be a particle of mass its motion should be controlled by Newton's first law which claims that its motion, and therefore, also its propagation direction are remaining unchanged, if no force is exerted on it. This consideration leads to a contradiction between fundamental physical laws:

- Theory of relativity ($E=mc^2$);
- Newton's first law;
- Maxwell's equations.

I this context it shall be stressed that this contradiction not only concerns the modes in an optical resonator but furthermore any electromagnetic wave. A solution of this problem could be found, if one assumes that the mass of light does not follow the propagating energy. However, since it has been proven by many experiments that the direction of propagating light changes under the influence of gravity, which according to the theory of general relativity acts on the mass of light, it must be assumed that energy and mass are propagating together in the same direction. Therefore, the question arises, why does the propagating relativistic mass density follow the propagating energy density? Which interaction takes place between the propagating mass and the propagating energy?

To investigate this interaction in more detail we consider that a propagating wave can be thought to be made up by a bundle of thin bent channels whose walls follow the lines of the propagating Poynting vector. In Fig. 2 these walls are shown by green lines, and the mass propagating within the channels is visualized by blue arrows.

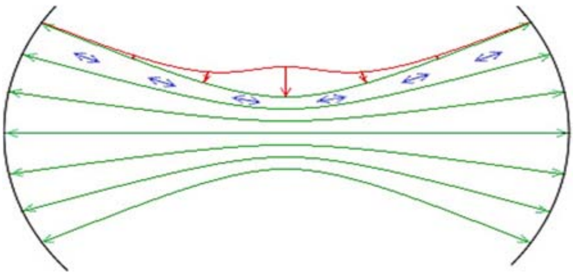

Fig. 2. Subdivision of a wave into a bundle of thin channels whose walls follow the green lines of the propagating Poynting vector. The blue arrows symbolize the propagating mass density. The red arrows symbolize the force exerted on the mass density. The distance between the red and the topmost green line symbolizes, how this force changes along the propagation direction as shown by Eq. (24) for a Gaussian wave.

Due to the momentum change of the mass density propagating along these channels, the mass of the light is compressed like a compressible fluid, when the wave propagates from the right mirror to the waist, and is expanding again, after the wave has passed through the waist. It must therefore be assumed that a force opposite to the change of the momentum of the mass density is exerted on the propagating mass density as shown by the red arrows in Fig. 2. In principle, this force seems to be comparable with the force exerted on the body of a motorcyclist driving along a narrow curve. Therefore, if a small particle of the propagating mass is considered, it can be concluded that a force is exerted on this particle opposite to the infinitesimal momentum change of this particle during an infinitesimal propagation step. Since this momentum change must be proportional to the directional change of the propagating energy density, it can furthermore be concluded that the transverse force exerted on the particle must be proportional to the negative value of the directional change of the normalized Poynting vector versus an infinitesimal propagation step.

To derive an expression for the infinitesimal directional change of the Poynting vector we consider two equiphase surfaces $\Phi_i(r,z_i)$ of a propagating wave with infinitesimal distance as shown in Fig. 3, where for the sake of simplicity rotational symmetry of the wave is assumed. The z_i are the points where the $\Phi_i(r,z_i)$ intersect the optical axis. The red arrows symbolize the normalized Poynting vectors erected in the points r_i located on the Φ_i. The points r_i as well as z_i are assumed to move into each other in the infinitesimal limit. The time, which the phase front Φ_1 takes to propagate into Φ_2, shall for further use be designated by Δt. Since according to the above arguments the direction of the Poynting vector must be assumed to be in alignment with the direction of the momentum of the propagating mass density, it can be concluded that the change of the momentum of a small particle of mass during the time Δt is proportional to the change of the normalized Poynting vector during Δt. In this way, it can furthermore be concluded that the transverse force exerted on a small particle of mass is proportional to the negative value of the differential quotient obtained after dividing the infinitesimal change of the normalized Poynting vector by Δt according to the following expression

$$\vec{K}(r,z) \propto - \lim_{z_1 \to z_2, r_1 \to r_2, \Delta t \to 0} \frac{1}{\Delta t}\left[\vec{S}_N(r_2,z_2) - \vec{S}_N(r_1,z_1)\right] \quad (1)$$

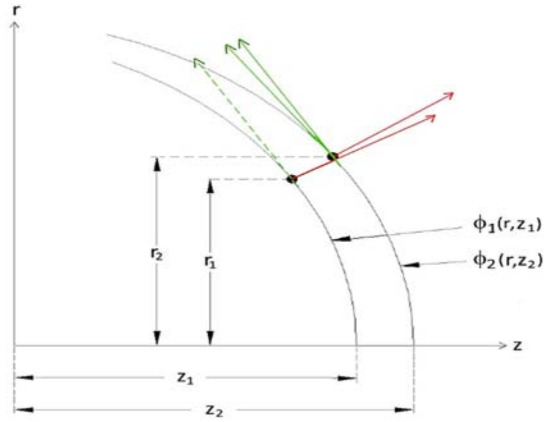

Fig. 3. Visualization of two phase fronts Φ_1 and Φ_2 intersecting the z axis at z_1 and z_2.

In Eq. (1) $\mathbf{S}_N(r_i,z_i)$ designates the normalized Poynting vector erected on $\Phi(r,z_i)$ in the point r_i as visualized by the red arrows in Fig. 3. Since the particle of mass is propagating with the speed of light c, its momentum is given by $\tilde{M}c$, if $\tilde{M}$ is its mass. Since according to Newton's second law a momentum change divided by a time interval describes a force, it can furthermore be concluded that the force exerted on the particle is given by

$$\vec{K}(r,z) = -\tilde{M}c^* \lim_{z_1 \to z_2, r_1 \to r_2, \Delta t \to 0} \frac{1}{\Delta t}\left[\vec{S}_N(r_2,z_2) - \vec{S}_N(r_1,z_1)\right] \quad (2)$$

In order to derive an expression for Δt, we look for two points r_i for which $\mathbf{S}(r_1)$ and $\mathbf{S}(r_2)$ are in alignment with each other. Under this condition the energy density of the electromagnetic wave propagates exactly perpendicularly to the Φ_i according to the orientation of the Poynting vector. Therefore, we obtain under this condition

$$\Delta t = \frac{\Phi(r_2,z_2) - \Phi(r_1,z_1)}{c} \quad (3)$$

keeping in mind that this relation is only valid, if the above condition is met. It may be argued that Eq. (3) may deliver different results for Δt, if the above condition is met by more than one pair of points r_i. However, this is not the case, since, due to the constant speed of light, Eq. (3) always delivers the same result for Δt under the above condition. For rotational symmetric waves, the above condition is met for $r_1 = r_2 = 0$. Therefore, we obtain in this case

$$\Delta t = \frac{z_2 - z_1}{c} = \frac{\Delta z}{c} \tag{4}$$

This delivers for the force exerted on a particle of mass $\tilde{M}$

$$\vec{K}(r,z) = -\tilde{M}c^2 \lim_{r_1 \to r_2, \Delta z \to 0} \frac{1}{\Delta z}\left[\vec{S}_N(r_2, z_2) - \vec{S}_N(r_1, z_1)\right] \tag{5}$$

So far arguing is based on terms of classical physics combined with the theory of relativity. Therefore, the question arises what is the physical meaning of a small particle of mass propagating with an electromagnetic wave as considered above. To answer this question, we take into account that the propagating energy is subdivided into quanta of energy called photons. Therefore, since the above argumentation does not require defining the mass of the small particle exactly, it can be assumed that the mass of the small particle is represented by the relativistic mass of the photon. Thus, it should be possible to assume that the force exerted on the particles of mass can be interpreted as a force exerted on the photons, even though it may not be possible to assume that the photons exactly propagate within channels with infinitesimal cross section. Thus, after replacing the mass $\tilde{M}$ by the relativistic mass M of the photon, we obtain

$$\vec{K}(r,z) = -E_{ph} * \lim_{r_1 \to r_2, \Delta z \to 0} \frac{1}{\Delta z}\left[\vec{S}_N(r_2, z_2) - \vec{S}_N(r_1, z_1)\right], \tag{6}$$

where E_{ph} is the energy of the photon given by

$$E_{ph} = Mc^2 = \frac{hc}{\lambda}, \tag{7}$$

where λ and c are the wavelength and the speed of light in a vacuum, respectively. M=h/(cλ) is the relativistic mass of the photon. Eq. (6) shows that the force K(r,z) exerted on the photon is proportional to its energy E_{ph}. Eq. (6) can be transformed furthermore, if we take into account that the directional change of the normalized Poynting vector $\mathbf{S}_N$(r,z) versus the z coordinate is identical with the negative value of the change of the tangents to the phase fronts Φ(r,z) versus the r coordinate, as visualized by the green arrows in Fig. 3. This allows to transform Eq. (6) into

$$\vec{K}(r,z) = E_{ph} * \lim_{r_1 \to r_2, \Delta z \to 0} \frac{1}{\Delta z}\left[\frac{d\Phi(r_2, z_2)}{dr} - \frac{d\Phi(r_1, z_1)}{dr}\right] \tag{8}$$

The use of the coordinates r,z in the above equations does not mean that the photon is located at r,z, it only means that the force **K** is exerted on the photon, if the photon can be found at the position described by r,z. This is in agreement with the quantum mechanical assumption that a Coulomb force is exerted on an electron, if the latter can be found at a certain distance from the atomic nucleus. Without this assumption, no quantum mechanical potential, and therefore not even the Schrödinger equation describing the hydrogen atom could have been derived.

The remaining question, if the force exerted on the photon is a real force may be answered as follows. If one replaces in Eq. (2) $\tilde{M}c$ by Mc one obtains that the force **K**(r,z) exerted on the photon is equal to the momentum of the photon multiplied with the infinitesimal directional change of the normalized Poynting vector versus the infinitesimal time step Δt. Therefore, **K**(r,z) physically represents a force, though it may not be possible to conclude that a photon, which is found at a certain position r,z, changes the direction of its momentum at this position. However, it seems to be necessary to assume that an overall continuous directional change of the momentum of the photons propagating with an electromagnetic wave takes place, since otherwise the photons would not follow the propagating wave, and would spread out from the wave due to their relativistic mass. Therefore, it seems to be possible to conclude that **K**(r,z) represents a force which causes a directional change of the momentum of the photons, and in this way, forces the photons to follow the electromagnetic wave. But since on the other hand the photons make up the electromagnetic wave, it may be necessary to assume that some kind of self-interaction of the photons takes place.

3. Derivation of a Potential and a Schrödinger Equation Describing the Transverse Motion of the Photon

Integration of the negative value of the force **K**(r,z) given by Eq. (8) over r along the curvature of the phase front Φ(r,z) shows that the photon is moving within a transverse potential given by

$$V(r,z) = E_{ph} * \int_0^r \lim_{r_1 \to r_2, \Delta z \to 0} \frac{1}{\Delta z}\left[\frac{d\Phi(r_1, z_1)}{dr} - \frac{d\Phi(r_2, z_2)}{dr}\right] dr \tag{9}$$

This equation shows that the transverse motion of the photon is described by a potential like the motion of the electron. It can be therefore assumed that its transverse motion is also described by a Schrödinger equation like the motion of the electron except for the difference that the mass of the electron is replaced by the relativistic mass M of the photon. Thus, it seems to be possible to conclude that the transverse motion of the photon is described by the following Schrödinger equation

$$\left[\frac{\hbar^2}{2M}\Delta_\perp + E(z) - V(\mathrm{r},z)\right]\chi(\mathrm{r},z) = 0\,, \qquad (10)$$

where r and z are not Cartesian coordinates in the usual sense, since z describes the point where the phase front Φ(r,z) intersects the optical axis. From Eq. (10) it turns out that the Schrödinger equation not only describes the motion of a particle with rest mass, but also describes the motion of the photon which only has a relativistic mass. This furthermore shows that the transverse motion of the photon is described by matter wave theory as introduced by de Broglie and Schrödinger almost hundred years ago to describe the motion of the electron. Therefore, the photon exhibits wave-particle duality like the electron, but even more. On the one hand, its motion along propagation direction is described by wave optics, and on the other hand, its transverse quantum mechanical motion is described by matter wave theory.

The eigensolutions χ(x,y,z) of the above Schrödinger equation allow to compute the probability density $|\chi(x,y,z)|^2$ of the photons propagating with an electromagnetic wave. This result shows how the photon can be assumed to be embedded into the propagating electromagnetic wave with its relativistic mass. This seems to be in contradiction to the statement of David Bohm mentioned in the introduction.

4. Verification of the Eqs. (8), (9) and (10) for the Case of the Plane, the Spherical, and the Gaussian Wave

Since in case of a plane wave dΦ(r,z)/dr vanishes for all values of r and z, the latter obviously also holds for the potential V(r,z). Therefore, since for a vanishing potential the transverse motion of the photon is not confined, no quantum mechanical eigenfunctions with a confined transverse extension exist. This is in agreement with the wave optics result that plane waves are not described by eigenmodes with confined transverse extension. The same holds for the spherical wave. Since in this case the Poynting vectors **S**(r_1,z_1) and **S**(r_2.z_2) are in alignment with each other for all r_i and z_i, the relation dΦ(r_1,z_1)/dr = dΦ(r_2,z_2)/dr holds for all r_i, z_i with the consequence that also in this case the potential vanishes.

However, the Eqs. (8), (9) and (10) also prove true for the case of a Gaussian wave. To shows this, we take into account that the phase of a wave front as shown in Fig. 3 is in the case of a Gaussian wave according to Eq. (4.7.7) in [6] given by

$$\Phi(\mathrm{r},z) = z - \frac{r^2}{2R(z)} \qquad (11)$$

as explained in more detail in [3]. Here R(z) is the radius of curvature of the phase front of a Gaussian wave which according to Eq. (17.5) in [7] is given by

$$R(z) = z + \frac{z_R^2}{z}\,, \qquad (12)$$

where z_R is the Rayleigh range which according to Eq. (17.4) in [7] is given by $z_R=\pi w_0^2/\lambda$, where w_0 is the spot size at the beam waist of the Gaussian wave. Therefore, after inserting Eq. (11) into Eq. (8), and carrying through the differentiation versus r we obtain

$$\vec{\mathrm{K}}(\mathrm{r},z) = E_{ph} * \lim_{r_1\to r_2,\Delta z\to 0}\frac{1}{\Delta z}\left[\frac{\mathrm{r}_1}{R(z_1)} - \frac{r_2}{R(z_2)}\right] \qquad (13)$$

As shown in [3], this expression can be transformed into

$$\vec{\mathrm{K}}(\mathrm{r},z) = -E_{ph} r\left(\frac{z_R}{z^2+z_R^2}\right)^2 \qquad (14)$$

To compute the potential, -K(r,z) has to be integrated along the curvature of the phase front. However, in paraxial approximation this integration can be carried through directly along r. This delivers

$$V(r,z) = \frac{1}{2}M\omega_\perp^2(z)r^2\,, \qquad (15)$$

where $\omega_\perp$ is the frequency of the transverse quantum mechanical oscillation of the photon, and is according to Eq. (14) given by

$$\omega_\perp(z) = \frac{cz_R}{z^2+z_R^2} \qquad (16)$$

If the potential V(r,z), as given by Eq. (15), is inserted into the Schrödinger equation given by Eq. (10) the following expression

$$\left[\frac{\hbar}{2M}\Delta_\perp + E_{nm}(z) - \frac{1}{2}M\omega_\perp^2(z)(x^2+y^2)\right]\chi_{nm}(x,y,z) = 0 \qquad (17)$$

is obtained, where the coordinates x,y and z are used in the same way as r and z in Eq. (15). Eq. (17) is identical with the Schrödinger equation for the 2-dimensional harmonic oscillator. As well known, the eigenvalues $E_{nm}(z)$ of this Schrödinger equation are given by

$$E_{nm}(z) = \hbar\omega_\perp(z)(n+m+1) \qquad (18)$$

The eigensolutions of Eq. (17) are given by

$$\chi_{nm}(x,y,z) = \\ = \sqrt{\frac{2}{\pi}}\frac{1}{w_p(z)\sqrt{2^{n+m}n!m!}}H_n\left(\frac{\sqrt{2}x}{w_p(z)}\right)H_m\left(\frac{\sqrt{2}y}{w_p(z)}\right)\exp\left(-\frac{x^2+y^2}{w_p^2(z)}\right) \qquad (19)$$

with w_p^2 given by

$$w_p^2(z) = \frac{2\hbar}{M\omega_\perp(z)}, \tag{20}$$

where the subscript p refers to particle. According to Eq. (19), w_p represents the distance from the optical axis, where the value of the probability density of a photon in the ground level drops to $|\chi_{00}(x,y,z)|^2 = \exp(-2) * |\chi_{00}(0,0,z)|^2$. After inserting of $\omega_\perp$ according to Eq. (16) w_p^2 transforms into

$$w_p^2(z) = \frac{2\hbar z_R}{Mc}\left[1+\left(\frac{z}{z_R}\right)^2\right], \tag{21}$$

where the factor at the left side of the square bracket can be replaced by

$$w_p^2(0) = \frac{2\hbar}{Mc} z_R = \frac{\lambda}{\pi} z_R \tag{22}$$

This shows that the expression obtained for $w_p(0)$ is identical with the expression obtained by the use of paraxial wave optics for the Gaussian spot size w_0 at the beam waist as shown by Eq. (17.4) in [7]. Therefore, Eq. (21) can be transformed into

$$w_p(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}, \tag{23}$$

which is according to Eq. (17.5) in [7] identical with the expression for w(z) describing the z dependence of the spot size of a Gaussian wave. By the use of the Eq. (22) the expression for the force K(r,z) can be transformed as follows

$$\mathrm{K(r,z)} = -E_{ph} r\left(\frac{z_R}{z^2+z_R^2}\right)^2 = -E_{ph}\frac{w_0^4}{z_R^2}\frac{r}{w_p^4(z)} \tag{24}$$

Thus, the force exerted on photons in the fundamental mode, which can be found at a distance $w_p(z)$ from the axis, is given by

$$\mathrm{K}(\mathrm{w}_p(\mathrm{z}),\mathrm{z}) = -E_{ph}\frac{w_0^4}{z_R^2}\frac{1}{w_p^3(z)} \tag{25}$$

This expression shows that $K(w_p(z),z)$ is strongest, when the mass density follows the narrow curve near the beam waist, as shown by the red arrows in Fig. 2 where the distance between the red line and the topmost green line visualizes the force $K(w_p(z),z)$. This force is becoming smaller with increasing distance from the waist, and disappears, when the beam is going over into a spherical wave as also shown in Fig. 2. Since in the latter case the mass density propagates along straight lines, no transverse force is exerted on the photons, as already discussed above.

Comparison of Eq. (19) with Eq. (16.60) in [7] shows that the probability density $|\chi_{nm}(x,y,z)|^2$ of the photon is in full agreement with the normalized local intensity provided by paraxial wave optics for a Gaussian mode of order n,m. In this way, the Eqs. (8), (9) and (10) could be verified for the case of a Gaussian wave. In the next section, it will be shown that also the Gouy phase shift [5] can be computed on the basis of the above results. This will provide an additional verification of the Eqs. (9) and (10).

5. Quantum Mechanical Computation of the Gouy Phase Shift for the Case of a Gaussian Wave

In the following, a computation of the Gouy phase shift [5] is given based on the above presented quantum mechanical particle picture. For this purpose, we consider the expression for the quantum mechanical expectation value of the square of the momentum in case of a 2-dimensional harmonic oscillator. This expression delivers for the expectation value of the square of the photon's transverse momentum

$$<\chi_{nm}\,|\,\hat{p}_\perp^2(z)\,|\,\chi_{nm}> = \hbar M\omega_\perp(z)(n+m+1) \tag{26}$$

By the use of the Eq. (16) for $\omega_\perp$ and by replacing M by $h/(c\lambda)$ the expression $M\omega_\perp$, used in this equation, can be transformed into $2\hbar/w^2(z)$ as shown in [3] in more detail. Insertion of this result into Eq. (26) delivers

$$<\chi_{nm}\,|\,\hat{p}_\perp^2(z)\,|\,\chi_{nm}> = \frac{2\hbar^2}{w^2(z)}(n+m+1) \tag{27}$$

For further purposes we consider now that the effective axial propagation constant for a finite beam can according to Eq. (3) in [8] be expressed as

$$\bar{k}_z(z) = \frac{<k_z^2>}{k} = k - \frac{<k_x^2>+<k_y^2>}{k} \tag{28}$$

In order to establish a relationship between the expectation value $<k_x^2>+<k_y^2>$ of the transverse part of the square of the propagation constant k and $<\chi_{nm}\,|\,\hat{p}_\perp^2(z)\,|\,\chi_{nm}>$ we take into account that the momentum of a freely propagating photon is given by $p=\hbar k$. This shows that the expression for the wave number k is obtained, if p is divided by $\hbar$. It can therefore be concluded that division of the expectation value $<\chi_{nm}\,|\,\hat{p}_\perp^2(z)\,|\,\chi_{nm}>$ by $\hbar^2$ delivers the expectation value $<k_x^2>+<k_y^2>$. In this way, we obtain

$$<k_x^2>+<k_y^2> = \frac{<\chi_{nm}\,|\,\hat{p}_\perp^2(z)\,|\,\chi_{nm}>}{\hbar^2} = \frac{2(n+m+1)}{w^2(z)} \tag{29}$$

Now we take into account that by the use of wave optics the following expression for the Gouy phase shift is obtained

$$\Phi_G = -\frac{1}{k}\int^z \left\{ < k_x^2 > + < k_y^2 > \right\} dz \qquad (30)$$

as shown by Eq. (4) in [8]. This expression delivers after insertion of Eq. (29) and carrying through the integration

$$\Phi_G = -(\mathrm{n} + m + 1)\arctan(z / z_R) \qquad (31)$$

in agreement with Eq. (20) in [8].

This result shows that the Gouy phase shift can be derived in full agreement with wave optics by the use of the quantum mechanical particle picture, and demonstrates that the Gouy effect can be equivalently understood as a wave optics as well as a quantum mechanical effect, even though the quantum mechanical description seems to provide a more straightforward understanding of some features of this effect. For instance the factor n+m+1 in Eq. (31) immediately follows from the quantum mechanical description of the 2-dimensional harmonic oscillator. Also the fact that the Gouy phase shift for a cylindrical wave is given by

$$\Phi_G = -(n + 1/2)\arctan(z / z_R) \qquad (32)$$

immediately follows from the fact that in this case the photon is moving within the potential of a 1-dimensional harmonic oscillator. Moreover, by the use of the particle picture the Gouy phase shift can easily be computed for an elliptic Gaussian beam. In this case the potential is given by

$$V(\mathrm{x,y},z) = V(\mathrm{x},z) + V(\mathrm{y},z) = \frac{1}{2}M\left[\omega_{\perp x}^2(z)x^2 + \omega_{\perp x}^2(z)y^2\right], \qquad (33)$$

where $\omega\perp_x$ and $\omega\perp_y$ are the transverse oscillation frequencies of the photon in x and y direction. This delivers

$$< k_x^2 > + < k_y^2 > = \frac{1}{\hbar^2} < \chi_{nm} \mid \hat{p}_\perp^2(z) \mid \chi_{nm} > = \frac{2n+1}{w_x^2(z)} + \frac{2m+1}{w_y^2(z)}, \qquad (34)$$

where $w_x(z)$ and $w_y(z)$, respectively, describe the z dependence of the spot size in the x-z and the y-z plane analogous to Eq. (23). For $w_x=w_y$ Eq. (34) transforms into Eq. (29).

These results provide, for the case of a Gaussian wave, an additional verification of the particle picture of the photon described by the Eqs. (9) and (10).

6. Explaining the Gouy Phase Shift as an Energetic Effect

However, the quantum mechanics particle picture not only allows to compute the Guoy Phase shift, it allows furthermore to explain the Guoy phase shift as an energetic effect. For this purpose we rewrite Eq. (26) by the use of Eq. (18) as follows

$$< \chi_{nm} \mid \hat{p}_\perp^2(z) \mid \chi_{nm} > = ME_{nm}(z) \qquad (35)$$

By the use of this equation Eq. (29) can be transformed into

$$< k_x^2 > + < k_y^2 > = \frac{ME_{nm}(\mathrm{z})}{\hbar^2} \qquad (36)$$

If we insert this expression into Eq. (28), we obtain

$$\bar{k}_z(z) = k\left(1 - \frac{ME_{nm}(z)}{k^2\hbar^2}\right) \qquad (37)$$

If we replace in this equation the relativistic mass M of the photon by $M=E_{ph}/c^2$, we obtain

$$\bar{k}_z(z) = k\left(1 - \frac{E_{ph}E_{nm}(z)}{c^2k^2\hbar^2}\right) \qquad (38)$$

If we take into account now that E_{ph} can be expressed as

$$E_{ph} = ck\hbar \qquad (39)$$

we obtain

$$\bar{k}_z(z) = k\left(1 - \frac{E_{nm}(z)}{E_{ph}}\right) = \frac{k}{E_{ph}}\left[E_{ph} - E_{nm}(z)\right] = \frac{1}{\hbar c}\left[E_{ph} - E_{nm}(z)\right] \qquad (40)$$

This equation can be rewritten as

$$\hbar c\bar{k}_z(z) = E_{ph} - E_{nm}(z) \qquad (41)$$

This shows that $\hbar c\bar{k}_z(z)$ is equal to the difference between the total energy E_{ph} of the photon and the energy eigenvalues $E_{mn}(z)$ of the Schrödinger equation which increase with decreasing z, since $\omega\perp$ increases according to Eq. (16). It can therefore be concluded that part of the total energy of the photon is transformed into the energy of the transverse quantum mechanical motion of the photon, and another part is transformed into $\hbar c\bar{k}_z(z)$. Since according to this result $\hbar c\bar{k}_z(z)$ represents a real energy, it must be assumed that also the effective axial propagation constant $\bar{k}_z(z)$ represents a real propagation constant. This leads to the conclusion that

$$\lambda_{nm}(z) = \frac{2\pi}{\bar{k}_z(z)} = \frac{hc}{\hbar c\bar{k}_z(z)} = \frac{hc}{E_{ph} - E_{nm}(z)} \qquad (42)$$

represents the real local wave length of the photon at the position z. According to this conclusion $\lambda_{nm}(z)$ increases inversely proportional to $\bar{k}_z(z)$, and therefore describes the Gouy phase shift in agreement with wave optics. But this conclusion also shows that $\lambda_{nm}(z)$ increases inversely proportional to energy difference $E_{ph} - E_{nm}(z)$ which decreases with decreasing z. This seems to show that the deeper physical reason for Gouy phase shift consists in the fact that the energy of the photon is increasingly converted into its transverse quantum mechanical motion when the photon approaches the focus. This allows to conclude that the Gouy phase shift is an energetic effect. Therefore, it seems to be reasonable to designate the energy $\hbar c\bar{k}_z(z)$ as Gouy energy, and to introduce for this energy the term $E_{G,nm}(z)$. Based on this result Eq. (41) can be rewritten as

$$E_{ph} = E_{G,nm}(z) + E_{nm}(z), \tag{43}$$

which shows that the total energy E_{ph} is the sum of an axial part described by $E_{G,nm}(z)$, and the energy eigenvalues $E_{mn}(z)$ of the Schrödinger equation.

Since $E_{G,nm}(z)$ vanishes for $E_{nm}(z) = E_{ph}$, $\lambda_{nm}(z)$ goes to infinity under this condition. Therefore, $E_{nm}(z) < E_{ph}$ must be considered to be a limiting condition for E_{nm}. The condition $E_{nm}(z) < E_{ph}$ transforms, after replacing $E_{nm}(z)$ according to Eq. (18) and E_{ph} according to Eq. (7), into

$$\hbar\omega_{\perp}(z)(n+m+1) < \frac{hc}{\lambda}, \tag{44}$$

which after replacing $\omega_{\perp}(z)$ according to Eq. (16) transforms into

$$\frac{z_R}{z^2 + z_R^2}(n+m+1) < \frac{2\pi}{\lambda} \tag{45}$$

This equation transforms for z=0 into

$$n+m+1 < \frac{2\pi}{\lambda} z_R = 2\left(\frac{\pi w_0}{\lambda}\right)^2 \tag{46}$$

This delivers for the spot size w_0 at the beam waist the condition

$$w_0 > \frac{\lambda}{\pi}\sqrt{\frac{n+m+1}{2}} \tag{47}$$

If we replace in this equation the ">" sign by the "=" sign we obtain for n=m=0

$$w_0 = \frac{\lambda}{\sqrt{2}\pi} \tag{48}$$

This delivers the relation

$$\omega_{\perp}(0) = \frac{c}{z_R} = \frac{2\pi c}{\lambda} = \omega \tag{49}$$

This result shows that under the condition $E_{nm}(z) = E_{ph}$ the frequency $\omega_{\perp}$ of transverse quantum mechanical oscillation of the photon is becoming equal to the frequency ω of the initially incoming electromagnetic wave. Therefore, in this case the oscillation of the wave is in the focus totally transformed into the transverse quantum mechanical oscillation of the photon. Simultaneously, the effective axial propagation constant $\bar{k}_z$ vanishes according to Eq. (40), and the local wave length $\lambda_{nm}(z)$ goes to infinity according to Eq. (42).

Concerning the Eqs. (46-48) it shall be mentioned that there is mathematical agreement between Eq. (48) and Eq. (30) given in [9] which according to the considerations given in [9] describes the fundamental mode radius. Eq. (47) can be therefore considered as a generalization of Eq. (30) in [9] for higher order Gaussian modes. This shows once more that the particle provides results in agreement with wave optics as has been used in [9].

The fact that effective axial propagation constant $\bar{k}_z$ vanishes under the condition $w_0 = \lambda/\sqrt{2}\pi$ also can be derived from the wave optics description of the Gouy effect. If one inserts the relation given for $\langle k_x^2\rangle + \langle k_y^2\rangle$ by Eq. (10) in [8] into Eq. (28), which is identical with Eq. (3) in [8], one obtains

$$\bar{k}_z(0) = \frac{2\pi}{\lambda} - \frac{2\lambda}{2\pi w_0^2} = \frac{2\pi}{\lambda} - \frac{\lambda 2\pi^2}{\pi\lambda^2} = 0 \tag{50}$$

Since, as shown in [8], $\bar{k}_z$ is associated with the overall propagation phase $\Phi(z)$ through $\bar{k}_z = \partial\Phi(z)/\partial z$, the propagation phase $\Phi(z)$ stops to change at the beam waist for $\bar{k}_z(z=0) = 0$, with the consequence that the local wave length $\lambda_{nm}(z)$, which according to Eq. (42) is inversely proportional to $\bar{k}_z(z)$, goes to infinity. In this way, the wave optics consideration confirms the results obtained above by the use of the particle picture. However, the particle picture delivers a more comprehensive explanation, since it shows that the deeper physical reason for this effect consists in the fact that the energy E_{ph} of the photon is totally transformed into the energy $E_{nm}(z=0)$ of the transverse quantum mechanical motion of the photon for $\bar{k}_z(z) = 0$.

Concerning the result that the total energy E_{ph} of the photon is splitting into $E_{G,nm}+E_{nm}$ the question arises which energy has to be used in Eq. (1). To answer this question it shall be mentioned that the factor E_{ph} in Eq. (1) has initially been obtained as c(Mc) in Eq. (2). Since however the expectation value of the momentum of a harmonic oscillator vanishes, also the expectation value of the momentum of the transverse quantum mechanical motion of the photon vanishes with the consequence that the total value Mc of the momentum of the photon remains unchanged. Therefore, also the factor E_{ph} in Eq. (6) remains unchanged. Since the momentum of the photon remains unchanged, also the radiation pressure remains unchanged in the focal region which is important, since both quantities are used to describe the optical tweezers [10-11].

7. Summary and Conclusions

As well-known from the propagation of laser beams the energy density of an electromagnetic wave is usually not propagating along a straight line. Therefore, the question has been considered, why the relativistic mass density of an electromagnetic wave follows the propagating energy density, even if the Poynting vector changes its direction. Since this change of the propagation direction of the mass density is in contradiction with Newton's first law, it is assumed that a transverse force is exerted on the photons. This assumption leads to the result that the photon is moving within a transverse potential which allows to describe the transverse quantum mechanical motion of the photon by a Schrödinger equation which is identical with the Schrödinger equation describing the motion of the electron, except that the mass of the electron is replaced by the relativistic mass of the photon [3-4]. This result shows that the transverse motion of the photon is described by matter wave theory, even though the photon has no rest mass. The eigensolutions $\chi(x,y,z)$ of the obtained Schrödinger equation allow to compute the probability density $|\chi(x,y,z)|^2$ of a photon propagating with an electromagnetic wave. In case of Gaussian waves the obtained Schrödinger equation is identical with the Schrödinger equation of the 2-dimensional harmonic oscillator [3-4]. These results seem to be in contradiction to the statement of David Bohm mentioned in the introduction.

The question, if the force exerted on the photons represents a real force, has been considered at the end of Sect. 1. Eq. (2) shows after replacing $\tilde{M}c$ by the momentum Mc of the photon that the force described by **K**(r.z) is equal to the momentum of the photon multiplied with the infinitesimal directional change of the normalized Poynting vector versus a time step Δt describing an infinitesimal propagation step of the wave. Therefore, since an infinitesimal momentum change versus an infinitesimal time step represents a force, the modified right side of Eq. (2) physically represents a force, though it may not be possible to conclude that a photon, which is found at a certain position r,z, changes the direction of its momentum at this position. Since however nevertheless an overall continuous directional change of momentum of the photons must take place, which forces them to follow the electromagnetic wave, it seems that **K**(r.z) represents the force which causes this momentum change.

Another interesting aspect of the force exerted on the photon consists in the fact that, in case of a Gaussian wave, a ray bouncing between two virtual phase fronts with infinitesimal distance carries through a sinusoidal motion which according to Eq. (48) in [1] is described by $r=A\cos(\omega_\perp t)$, and therefore oscillates with a frequency $\omega_\perp$ identical with the frequency of the quantum mechanical transverse oscillation of the photon given by Eq. (16). Therefore, since a ray of infinitesimal length can be considered to represent the classical equivalent to the photon, it follows that the transverse motion of this classical equivalent particle is described by the same potential which describes the motion of the photon with the difference that the photon obeys the laws of quantum mechanics. It can be therefore concluded that the same force, which is exerted on the classical equivalent particle, is also exerted on the photon. This shows that the force derived above by considering the directional momentum change of the photon, also can be derived from a geometric optics consideration.

The results described by the Eqs. (8), (9) and (11) have been verified for the plane, the spherical and the Gaussian wave. In the latter case the result obtained for the probability density $|\chi(x,y,z)_{nm}|^2$ of the photon is in full agreement with the normalized local intensity of the Gaussian modes of order n,m obtained by the use of paraxial wave optics. An additional verification has been obtained by considering the Guoy phase shift [5]. It could be shown that in case of a Gaussian wave a mathematical expression describing the Guoy phase shift can be derived from the particle picture which is in full agreement with the expression derived by the use of wave optics. Therefore, the Guoy phase shift can be equivalently understood as a wave optics as well as a quantum mechanical effect.

However, it could be shown that the quantum mechanics particle picture not only allows to compute the Guoy phase shift, it seems moreover to allow to explain the Guoy phase shift as an energetic effect. For this purpose the Schrödinger equation is used to compute the expectation value of the square of the photon's transverse momentum. This allows to express the effective axial propagation constant $\bar{k}_z(z)$ of the wave by $\bar{k}_z(z) = \left[E_{ph} - E_{nm}(z)\right]/\hbar c$ where E_{ph} is the total energy of the photon, and the $E_{nm}(z)$ are the energy eigenvalues of the Schrödinger equation. Since according to this result $\hbar c\bar{k}_z(z)$ represents a real energy, also the effective axial propagation constant $\bar{k}_z(z)$ must be assumed to represent a real propagation constant. This leads to the conclusion that $\lambda_{nm}(z) = 2\pi/\bar{k}_z(z) = hc/\left(E_{ph} - E_{nm}(z)\right)$ represents the real local wave length of the photon at the position z. According to this conclusion, $\lambda_{nm}(z)$ increases inversely proportional to $\bar{k}_z(z)$ with decreasing z, and therefore, describes z dependence of the Gouy phase shift in agreement with wave optics. But this conclusion also shows that $\lambda_{nm}(z)$ increases inversely proportional to the energy difference E_{ph}-$E_{nm}(z)$ which decreases with decreasing z. This shows that the deeper physical reason for Gouy phase shift seems to consist in the fact that the energy of the photon is increasingly converted into its transverse quantum mechanical motion when the photon approaches the focus. This explains the Gouy phase shift as an energetic effect. Based on this result the proposal has been made to designate the energy $\hbar c\overline{k}_z(z)$ as Gouy energy, and to introduce for this energy the term $E_{G,nm}(z)$. This delivers for the total energy of the photon the expression $E_{ph} = E_{G,nm}(z) + E_{nm}(z)$.

Since according to these results only $E_{Gnm}(z)$ changes when photon approaches the focus but not its total energy E_{ph}, it follows that the wave length of the photon as measured by a spectroscopic device is λ=hc/E_{ph} and not $\lambda_{nm}(z)$, because the measurement process changes the structure of the propagating wave with the consequence that the change of the wave length disappears together with the Gouy phase shift.

Since $E_{G,nm}(z)$ decreases, when the photon approaches the focus, and is transformed into $E_{nm}(z)$, which simultaneously increases by the same amount, it can be concluded that the wave structure of the photon is becoming less important compared with its particle property near the focus. This result may provide new aspects concerning the theoretical description of the optical tweezers [10-11].

Since a close relationship exists between the Gouy phase shift and the reduced effective axial propagation constant in a waveguide as described in [7] Sect. 17.4, it can be expected that the reduction of the propagation constant in a waveguide can be explained in the same way as the Gouy phase shift as an energetic effect induced by a partial transfer of the photon's energy to the transverse quantum mechanical motion of the photon, as will be described in more detail in a subsequent paper [12].

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